Hedging default risks of CDOs in Markovian contagion models

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AFFI 2008 Annual Meeting
21 May 2008

Presentation related to paper:
Hedging default risks of CDOs in Markovian contagion models (2007)
Joint work with Jean-Paul Laurent and Jean-David Fermanian
Available on www.defaultrisk.com
In interest rate or equity markets, pricing is related to the cost of the hedge
- ex: Black-Scholes pricing model, local volatility model

In credit markets, pricing is disconnect from hedging
- ex: The standard pricing model for CDO tranches does not rely on a replication argument

Need to relate pricing and hedging

In defaultrisk.com
- More than 1000 papers
- About 10 papers deal with hedging issues
Purpose of the presentation

- Focus on very specific aspects of default and credit spread risk
- Under which the market for CDO tranches is complete
  - CDO tranches can be perfectly replicated by dynamically trading CDS (Credit Default Swaps)

Overlook of the presentation

- Standardized CDO tranches
- Tree approach to hedging defaults
  - Analogue of the local volatility model of Dupire (1994) or Derman & Kani (1994) for credit portfolio derivatives
- Results and comments
  - Hedging strategies obtained from a tree calibrated on market data
  - Comparison with market practice
What is a standardized CDO tranche?

- Bilateral contract between a buyer of protection and a seller of protection

- The reference entity can be either
  - Credit Default Swap Index (Itraxx Europe, CDX North America)
  - Tranche or Portion of the aggregate loss associated with the Index
Standardized CDO tranches

What does tranche mean?

- Credit portfolio of $n$ reference entities
- $\tau_1, \ldots, \tau_n$ default times
- $\delta$ recovery rate
- Key drivers of CDO tranche cash-flows:

$$N_t = \sum_{i=1}^{n} 1_{\{\tau_i \leq t\}}$$

$$L_t = \frac{1 - \delta}{n} N_t$$

$$L_t^{[a,b]} = (L_t - a)^+ - (L_t - b)^+$$
Tree approach to hedging defaults

• We will start with two names only
  – Building a risk neutral tree of default states
  – Computation of prices along the tree for zero coupon CDO tranches

• Multiname case: homogeneous Markovian model
  – Building of risk-neutral tree for the aggregate loss
  – Computation of dynamic deltas

• Technical details can be found in the paper:
  – “hedging default risks of CDOs in Markovian contagion models”
Some notations:

- $\tau_1, \tau_2$ default times of counterparties 1 and 2,
- $H_t$ available information at time $t$,
- $Q$ risk neutral probability,
- $\alpha_1, \alpha_2$ : (risk neutral) default intensities:
  $$Q\left[\tau_i \in [t,t+dt] \mid H_t\right] = \alpha_i \, dt, \ i=1,2$$

Assumption of « local » independence between default events

- Probability of name 1 and name 2 defaulting altogether:
  $$Q\left[\tau_1 \in [t,t+dt], \tau_2 \in [t,t+dt] \mid H_t\right] = \alpha_1 \, dt \times \alpha_2 \, dt \text{ in } (dt)^2$$
- Local independence: simultaneous joint defaults can be neglected
Building up a tree – static case:

- Four possible states: \((D,D)\), \((D,ND)\), \((ND,D)\), \((ND,ND)\)
- Under no simultaneous defaults assumption \(p_{(D,D)} = 0\)
- Only three possible states: \((D,ND)\), \((ND,D)\), \((ND,ND)\)
- Identifying risk neutral tree probabilities:

\[
\begin{aligned}
\alpha_1 dt & \quad (D, ND) \\
\alpha_2 dt & \quad (ND, D) \\
1 - (\alpha_1 + \alpha_2) dt & \quad (ND, ND)
\end{aligned}
\]

\[
\begin{align*}
\{ & \quad \frac{p_{(D,D)}}{p_{(D,ND)}} = \frac{p_{(D,D)}}{p_{(D,ND)}} = p_{(D,.)} = \alpha_1 dt \\
& \quad \frac{p_{(D,D)}}{p_{(ND,D)}} = \frac{p_{(D,D)}}{p_{(ND,D)}} = p_{(.,D)} = \alpha_2 dt \\
& \quad p_{(ND,ND)} = 1 - p_{(D,.)} - p_{(.,D)}
\end{align*}
\]
Dynamic case:

\[
\begin{align*}
\alpha_1 \, dt & \quad (D, ND) \\
\alpha_2 \, dt & \quad (ND, D) \\
1 - (\alpha_1 + \alpha_2) \, dt & \quad (ND, ND)
\end{align*}
\]

- \( \lambda_2 \) intensity of name 2 after default of name 1
- \( \lambda_1 \) intensity of name 1 after default of name 2
- \( \pi_1 \) intensity of name 1 if no name defaults at period 1
- \( \pi_2 \) intensity of name 2 if no name defaults at period 1

Change in default intensities due to contagion effects

- Usually, \( \pi_1 < \alpha_1 < \lambda_1 \) and \( \pi_2 < \alpha_2 < \lambda_2 \)
Computation of prices and hedging strategies by backward induction

- Start from period 2, compute
  - price at period 1 for the possible nodes
  - hedge ratios in CDS on name 1 and in CDS on name 2 at period 1
- Start from period 1, compute
  - price at time 0
  - hedge ratios in CDS 1,2 at time 0
Tree approach to hedging defaults

- Example: zero coupon CDO tranchelets
  - Recovery rate=0, default free interest rate=0, maturity 2
  - Aggregate loss at time 2 can be equal to 0,1,2
    - Equity tranche contingent on no defaults
    - Mezzanine tranche: one default
    - Senior tranche: two defaults

- Senior Tranche:

\[
\begin{align*}
\alpha_1 dt & \quad (D, ND) \\
\frac{\alpha_1 dt \times \lambda_2 dt + \alpha_2 dt \times \lambda_1 dt}{\text{up-front premium default leg}} & \quad (ND, D) \\
\alpha_2 dt & \quad (ND, ND) \\
1 - (\alpha_1 + \alpha_2) dt & \quad (ND, ND)
\end{align*}
\]

\[
L_t = N_t
\]
• Mezzanine Tranche
  – Time pattern of default payments
  \[
  \alpha_1 dt + \alpha_2 dt + \left(1 - \left(\alpha_1 + \alpha_2\right)dt\right)\left(\pi_1 + \pi_2\right)dt
  \]
  
  – Possibility of taking into account discounting effects
  – The timing of premium payments
  – Computation of dynamic deltas with respect CDS on names 1,2
In theory, one could also derive dynamic hedging strategies for index CDO tranches.

- Numerical issues: large dimensional, non recombining trees
- Homogeneous Markovian assumption is very convenient

- Default intensities at a given time \( t \) only depend upon the current number of defaults \( N(t) \): 
  \[ \alpha_i(t, N(t)) \]
- Intensities at time 0 (no defaults) \( \alpha_1 = \alpha_2 = \alpha. (0,0) \)
- Intensities at time 1 (one default) \( \lambda_1 = \lambda_2 = \alpha. (1,1) \)
- Intensities at time 1 (no defaults) \( \pi_1 = \pi_2 = \alpha. (1,0) \)
Tree approach to hedging defaults

- Homogeneous Markovian tree

- If we have $N(1)=1$, one default at $t=1$
- The probability to have $N(2)=2$, two defaults at $t=2$...
- Is $\alpha_i (1,1)$ and does not depend on the defaulted name at $t=1$
- $N(t)$ is a Markov process
- Dynamics of the number of defaults can be expressed through a binomial tree
Tree approach to hedging defaults

- From name per name to number of defaults tree

\[
\begin{align*}
N(0) &= 0, \\
2\alpha \cdot (0,0) &\quad 1 - 2\alpha_1 (0,0) \\
N(1) &= 1, \\
2\alpha \cdot (0,0) &\quad 1 - 2\alpha_1 (0,0) \\
N(2) &= 1, \\
\alpha \cdot (1,1) &\quad 1 - \alpha \cdot (1,1) \\
N(2) &= 2, \\
\alpha \cdot (1,1) &\quad 1 - \alpha \cdot (1,1) \\
N(2) &= 0, \\
\alpha \cdot (1,0) &\quad 1 - \alpha \cdot (1,0) \\
N(2) &= 0, \\
\alpha \cdot (1,0) &\quad 1 - \alpha \cdot (1,0) \\
N(2) &= 0.
\end{align*}
\]
**Tree approach to hedging defaults**

- Easy extension to $n$ names
  - Individual intensity at time $t$ for $N(t)$ defaults: $\alpha_i(t, N(t))$
  - Number of defaults intensity: sum of surviving name intensities:
    \[
    \lambda(t, N(t)) = (n - N(t)) \alpha_i(t, N(t))
    \]

\[
N(0) = 0 \quad N(1) = 0 \quad N(2) = 0 \quad N(3) = 0
\]

\[
N(0) = 0 \quad N(1) = 1 \quad N(2) = 1 \quad N(3) = 1
\]

- $\alpha_i(0,0), \alpha_i(1,0), \alpha_i(1,1), \alpha_i(2,0), \alpha_i(2,1), \ldots$ can be easily calibrated
- on marginal distributions of $N(t)$ by forward induction.
**Tree approach to hedging defaults**

- Easy computation of CDO tranches and Index present values along the nodes of the tree using a backward induction.
What about the credit deltas?

- In a homogeneous framework, deltas with respect to CDS are all the same
- Perfect dynamic replication of a CDO tranche with a credit default swap index and the default-free asset
- Credit delta with respect to the credit default swap index
- $= \frac{\text{change in PV of the tranche}}{\text{change in PV of the CDS index}}$

$$\delta(t, N(t)) = \frac{CDO(t + 1, N(t) + 1) - CDO(t + 1, N(t))}{Index(t + 1, N(t) + 1) - Index(t + 1, N(t))}$$
Results and comments

- Calibration of the tree on a market base correlation structure
  - Number of names: 125
  - Default-free interest rate: 4%
  - 5Y Credit spreads: 20bps
  - Recovery rate: 40%

<table>
<thead>
<tr>
<th>3%</th>
<th>6%</th>
<th>9%</th>
<th>12%</th>
<th>22%</th>
</tr>
</thead>
<tbody>
<tr>
<td>18%</td>
<td>28%</td>
<td>36%</td>
<td>42%</td>
<td>58%</td>
</tr>
</tbody>
</table>

Table 6. Base correlations with respect to attachment points.

- Loss intensities with respect to the number of defaults
  - For simplicity, assumption of time-homogeneous loss intensities
  - Increase in intensities: contagion effects
  - Compare flat and steep base correlation structures

Figure 6. Loss intensities for the Gaussian copula and market case examples. Number of defaults are the x-axis.
Results and comments

- Dynamics of Credit Default Swap Index in the tree
  - In bps pa

<table>
<thead>
<tr>
<th>Nb Defaults</th>
<th>Weeks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>

- The first default leads to a jump from 19bps to 31 bps
- The second default is associated with a jump from 31 bps to 95 bps
- Explosive behavior associated with upward base correlation curve
Results and comments

- Dynamics of credit deltas ([0,3%] equity tranche):

<table>
<thead>
<tr>
<th>Nb Defaults</th>
<th>OutStanding Nominal</th>
<th>Weeks</th>
<th>0</th>
<th>14</th>
<th>56</th>
<th>84</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.00%</td>
<td></td>
<td>0.541</td>
<td>0.617</td>
<td>0.823</td>
<td>0.910</td>
</tr>
<tr>
<td>1</td>
<td>2.52%</td>
<td></td>
<td>0</td>
<td>0.279</td>
<td>0.510</td>
<td>0.690</td>
</tr>
<tr>
<td>2</td>
<td>2.04%</td>
<td></td>
<td>0</td>
<td>0.072</td>
<td>0.166</td>
<td>0.304</td>
</tr>
<tr>
<td>3</td>
<td>1.56%</td>
<td></td>
<td>0</td>
<td>0.016</td>
<td>0.034</td>
<td>0.072</td>
</tr>
<tr>
<td>4</td>
<td>1.08%</td>
<td></td>
<td>0</td>
<td>0.004</td>
<td>0.006</td>
<td>0.012</td>
</tr>
<tr>
<td>5</td>
<td>0.60%</td>
<td></td>
<td>0</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
</tr>
<tr>
<td>6</td>
<td>0.12%</td>
<td></td>
<td>0</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>7</td>
<td>0.00%</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- Deltas are between 0 and 1
- Gradually decrease with the number of defaults
  - **Equity tranche can be viewed as a short put position on the Index**
  - **Concave payoff, negative gammas**
- When the number of defaults is > 6, the tranche is exhausted
- Credit deltas increase in time
  - **Consistent with a decrease in time value**
Comparison of market deltas and tree deltas (at inception)
- Market delta computed under the standard Gaussian copula assumption
- Base correlation is unchanged when shifting spreads ("correlation-sticky deltas")
- Standard way of computing CDS index hedges in trading desks

<table>
<thead>
<tr>
<th>[0-3%]</th>
<th>[3-6%]</th>
<th>[6-9%]</th>
<th>[9-12%]</th>
<th>[12-22%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market deltas</td>
<td>27</td>
<td>4.5</td>
<td>1.25</td>
<td>0.6</td>
</tr>
<tr>
<td>Tree deltas</td>
<td>21.5</td>
<td>4.63</td>
<td>1.63</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Smaller equity tranche deltas in the tree model
- How can we explain this?
Results and comments

- Smaller equity tranche deltas in the tree model (cont.)
  - Default is associated with an increase in dependence

  Contagion effects

- Increasing correlation leads to a decrease in the PV of the equity tranche
- Recent market shifts go in favor of the contagion model

Figure 8. Dynamics of the base correlation curve with respect to the number of defaults. Detachment points on the $x$–axis. Base correlations on the $y$–axis.
Results and comments

- The current crisis is associated with joint upward shifts in credit spreads
  - Systemic risk
- And an increase in base correlations

Figure 9. Credit spreads on the five years iTraxx index (Series 7) in bps on the left axis. Implied correlation on the equity tranche on the right axis

- Tree deltas are well suited in regimes of fear
What do we learn from this hedging approach?

Thanks to stringent assumptions:
- credit spreads driven by defaults
- homogeneity
- Markov property

It is possible to compute a dynamic hedging strategy
- Based on the CDS index

That fully replicates the CDO tranche payoffs
- Model matches market quotes of liquid tranches
- Very simple implementation
- Credit deltas are easy to understand

Improve the computation of default hedges
- Since it takes into account credit contagion