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Dynamic Hedging of Portfolio Credit Risk in a Markov Copula Model

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Abstract We devise a bottom-up dynamic model of portfolio credit risk where instantaneous contagion is represented by the possibility of simultaneous defaults. Due to a Markovian copula nature of the model, calibration of marginals and dependence parameters can be performed separately using a two-steps procedure, much like in a standard static copula set-up. In this sense this solves the bottom-up top-down puzzle which the CDO industry had been trying to do for a long time. This model can be used for any dynamic portfolio credit risk issue, such as dynamic hedging of CDOs by CDSs, or CVA computations on credit portfolios.

Keywords Portfolio credit risk · Credit derivatives · Markov copula model · Common shocks · Dynamic hedging.

JEL Classification: G12, G13.

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1 Introduction

In this paper (see [1 - 3] for long versions with detailed proofs and numerics, respectively), we present a common shock model of portfolio credit risk where one can build a consistent picture of bottom up defaults that are also manageable in a top down aggregate loss space. In this sense this model solves the bottom-up top-down puzzle [4], which the CDO industry had been trying to do for a long time and basically failed. Then the CDO market died and the problem remained standing. The result of this paper, however, can be applied well beyond the space of, say, consistent valuation and hedging of CDSs and CDOs. In particular it is used in [5 - 7] for valuation and hedging of counterparty risk on credit derivatives.

The common shock aspect of our model is related to the work by Elouerkhaoui [8] and Brigo et al. [9 - 10] (see also [11 - 12]). The innovative breakthrough is a suitable decoupling property between the dependence structure and the individual names [13], so that the model can be jointly calibrated to single-name and portfolio data in two steps (as opposed to a global joint optimization procedures involving all the model parameters at the same time in the above references, which is untractable numerically).

2 Model of Default Times

In our model, defaults are the consequence of some "shocks" associated with groups of obligors. We define the following pre-specified set of groups

$$\mathcal{Y} = \{\{1\}, \ldots, \{n\}, I_1, \ldots, I_m\},$$

where $I_1, \ldots, I_m$ are subsets of $N = \{1, \ldots, n\}$, and each group I_j contains at least two obligors or more. The shocks are divided in two categories: the "idiosyncratic" shocks associated with singletons $\{1\}, \ldots, \{n\}$ can only trigger the default of name $1, \ldots, n$ individually, while the "systemic" shocks associated with multi-name groups $I_1, \ldots, I_m$ may simultaneously trigger the default of all names in these groups. Note that several groups I_j may contain a given name i, so that only the shock occurring first effectively triggers the default of that name. As a result, when a shock associated with a specific group occurs at time t, it only triggers the default of names that are still alive in that group at time t.

In the following, the elements Y of $\mathcal{Y}$ will be used to designate shocks and we let $\mathcal{I} = (I_l)_{1 \le l \le m}$ denote the pre-specified set of multi-name groups of obligors. Shock intensities $\lambda_Y(t, \mathbf{X}_t)$ will be specified later in terms of a Markovian factor process $\mathbf{X}_t$. Letting $\Lambda_t^Y = \int_0^t \lambda_Y(s, \mathbf{X}_s)ds$, we define

$$\tau_Y = \inf\{t > 0;\, \Lambda_t^Y > E_Y\}, \tag{1}$$

where the random variables E_Y are i.i.d. and exponentially distributed with parameter 1. For every obligor i we let

$$\tau_i = \min_{\{Y \in \mathcal{Y};\, i \in Y\}} \tau_Y, \tag{2}$$

which defines the default time of obligor i in the common shocks model. The model filtration is given as $\mathbb{F} = \mathbb{X} \vee \mathbb{H}$, the filtration generated by the factor process $\mathbf{X}$ and the point process $\mathbf{H} = (H^i)_{1 \le i \le n}$ with $H_t^i = \mathbb{1}_{\tau_i \le t}$.

This model can be viewed as a doubly stochastic (via the stochastic intensities Λ^Y) and dynamized (via the introduction of the filtration $\mathbb{F}$) generalization of the Marshall-Olkin model [12]. The purpose of the factor process $\mathbf{X}$ is to more realistically model diffusive randomness of credit spreads. Note that in [1], we construct the model the reverse way round, i.e. we first construct a suitable Markov process $(\mathbf{X}_t, \mathbf{H}_t)$ and then define the τ_i as the jump times of the H^i.

Figure 1 shows one possible defaults path in our model with $n = 5$ and

$$\mathcal{Y} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{4, 5\}, \{2, 3, 4\}, \{1, 2\}\}.$$

The inner oval shows which common-shock happened and caused the observed default scenarios at successive default times. At the first instant, default of name 2 is observed as the consequence of the idiosyncratic shock $\{2\}$. At the second instant, names 4 and 5 have defaulted simultaneously as a consequence of the systemic shock $\{4, 5\}$. At the fourth instant, the systemic shock $\{2, 3, 4\}$ triggers the default of name 3 alone as name 2 and 4 have already defaulted. At the fifth instant, default of name 1 alone is observed as the consequence of the systemic shock $\{1, 2\}$.

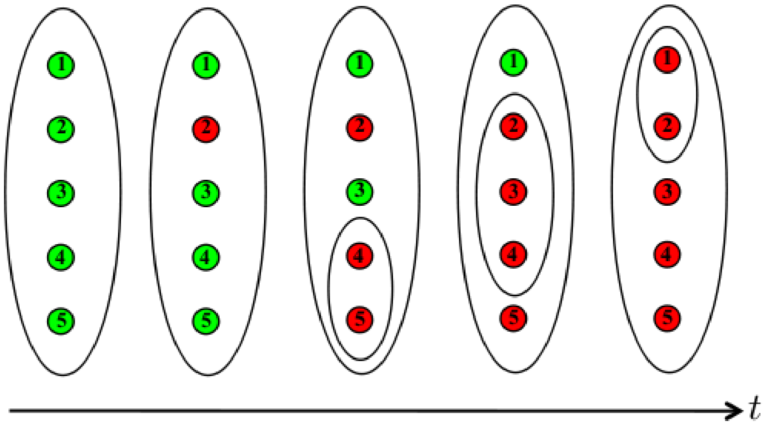

Fig. 1 One possible defaults path in a model with $n = 5$ and $\mathcal{Y} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{4, 5\}, \{2, 3, 4\}, \{1, 2\}\}$.

The set of obligors alive (resp. in default) at time t is denoted by $J_t = \text{supp}^c(\mathbf{H}_t)$ (resp. $I_t = \text{supp}(\mathbf{H}_t)$). For every $Y \in \mathcal{Y}$ and every set of non-negative constants $t, t_1, \ldots, t_n$, we define

$$\theta_t^Y = t \vee \max_{i \in Y \cap J_t} t_i$$

(with the convention that $\max \emptyset = 0$). Note that $Y \cap J_t$ in θ_t^Y represents the set of survivors in Y at time t. We also write

$$\Lambda_{s,t}^Y = \int_s^t \lambda_Y(u, X_u^Y) du, \;\; \lambda_t^i = \sum_{\{Y \in \mathcal{Y};\, i \in Y\}} \lambda_Y(t, \mathbf{X}_t).$$

The following result is proved in the Appendix.

Proposition 2.1 *For any fixed non-negative constants $t, t_1, \ldots, t_n$, we have:*

$$\mathbb{P}\left(\tau_1 > t_1, \ldots, \tau_n > t_n \mid \mathcal{F}_t\right) = \mathbb{1}_{\{t_i < \tau_i\,,\; i \in I_t\}} \mathbb{E}\left\{ \exp\big(- \sum_{Y \in \mathcal{Y}} \Lambda_{t, \theta_t^Y}^Y \big) \,\Big|\, \mathbf{X}_t \right\}. \tag{3}$$

In particular, for every obligor i and $t_i \geq t$,

$$\mathbb{P}(\tau_i > t_i \,|\, \mathcal{F}_t) = \mathbb{1}_{\{\tau_i > t\}} \mathbb{E}\left\{ \exp\big(- \int_t^{t_i} \lambda_s^i ds \big) \,|\, \mathbf{X}_t \right\}. \tag{4}$$

Thanks to formula (3), efficient convolution recursion procedures are available for pricing multi-name credit derivatives like CDO tranches (see [2] for more details). Thanks to formula (4) and under an additional affine structure postulated below on each individual pre-default intensity process λ^i, affine methodologies can be used to price single-name credit derivatives like CDSs, as CDS spreads can be derived from each individual survival distribution function. Additionally, the hedging issue can be dealt with numerically in this framework (see Section 3) since the pricing can be done conditionally on any given state of the dynamic model $(\mathbf{X}_t, \mathbf{H}_t)$. Another key interest of this framework is the fact that model parameters can be calibrated in two steps: individual λ^i-parameters are first calibrated to individual CDSs and the model dependence λ_I-parameters are then calibrated to CDO tranches (as opposed to a global joint optimization procedures involving all the model parameters at the same time, which would be untractable numerically). See [2] for the details. One can then consistently consider in this dynamic model issues like hedging CDO tranches by CDSs, or counterparty risk on credit derivatives.

Moreover, as announced above, in order to ensure the Markov consistency and Markov copula feature of the setup (see [13]), we assume further that every individual process λ^i is an affine process (in particular, a Markov process), as in either specification below. Consequently the conditioning with respect to $\mathbf{X}_t$ can be replaced by a conditioning with respect to λ^i_t in (4), hence exponential-affine methodogies for computing (4) follow.

Example 2.1 (i) (Deterministic group intensities). The idiosyncratic intensities $\lambda_{\{i\}}(t, \mathbf{X}_t)$ are affine, and the systemic intensities $\lambda_Y(t, \mathbf{X}_t)$ are deterministic functions of time, i.e. the functions $\lambda_Y(t, \mathbf{x})$ do not depend on $\mathbf{x}$, for $Y \in \mathcal{Y}$ that are not singletons.

(ii) (Extended CIR intensities). $\mathbf{X}_t = (X^Y_t)_{Y\in\mathcal{Y}}$ and for every $Y \in \mathcal{Y}$, $\lambda_Y(t, \mathbf{X}_t) = X^Y_t$, where X^Y_t is an extended CIR process

$$dX^Y_t = a(b_Y(t) - X^Y_t)dt + c\sqrt{X^Y_t}\, dW^Y_t, \tag{5}$$

for non-negative constants a, c (independent of Y) and a non-negative function $b_Y(t)$, and where the W^Y are independent standard Brownian motions.

In the second specification, affinity of λ^i (which is trivial in the first specification) arises from the fact that the SDE for the factors X^Y have the same coefficients except for the $b_Y(t)$. Thus, $X^i := \sum_{\{Y\in\mathcal{Y};\, i\in Y\}} X^Y$ satisfies the following extended CIR SDE:

$$dX^i_t = a(b_i(t) - X^i_t)dt + c\sqrt{X^i_t}dW^i_t, \tag{6}$$

for the function $b_i(t) = \sum_{\{Y \in \mathcal{Y};\, i \in Y\}} b_Y(t)$ and the Brownian motion

$$dW_t^i = \sum_{i \in Y} \frac{\sqrt{X_t^Y}}{\sqrt{\sum_{i \in Y} X_t^Y}} dW_t^Y .$$

3 Numerical Results

In this section we briefly discuss the calibration of the model and some few numerical results connected to the loss-distributions and the min-variance hedging of a CDO tranche by a portfolio of CDSs. See [2] for the details and [5 - 7] for further applications to counterparty risk modeling. Note that we only use piecewise-constant intensities $\lambda_Y(t)$ here (see [3] for numerical studies with stochastic intensities and random revoveries).

The model parameters are calibrated in two steps. First, the individual λ^i-parameters are calibrated to corresponding CDS spreads and we use piecewise-constant intensities on the time intervals $[0, 3]$ and $[3, 5]$ which yield perfect fits due to the bootstrapping algorithm.

In the second step we calibrate the common shock intensities $\lambda_{I_j}(t)$ for the m groups, which also are piecewise constant functions of time, so that $\lambda_{I_j}(t) = \lambda_{I_j}^{(1)}$ for $t \in [0, 3]$ and $\lambda_{I_j}(t) = \lambda_{I_j}^{(2)}$ for $t \in [3, 5]$ and for every group j. More specific, $\lambda_{I_j}(t) = \lambda_{I_j}^{(1)}$ are calibrated so that the five-year model spread $S_{a_l,b_l}(\boldsymbol{\lambda}) =: S_l(\boldsymbol{\lambda})$ will coincide with the corresponding market spread S_l^* for each tranche l. Thus, the parameters $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ are obtained according to

$$\boldsymbol{\lambda} = \operatorname*{argmin}_{\widehat{\boldsymbol{\lambda}}} \sum_l \left(\frac{S_l(\widehat{\boldsymbol{\lambda}}) - S_l^*}{S_l^*} \right)^2 \tag{7}$$

under the constraints that all elements in $\boldsymbol{\lambda}$ are non-negative and that $\boldsymbol{\lambda}$ satisfies the inequalities specified by the individual CDS calibrations (see [2] for the details) for every group I_l and in each time interval $[0, 3]$ and $[3, 5]$. We use Matlab in our numerical calculations and the objective function (7) is minimized by using the built in optimization routine `fmincon` together with the above mentioned constraints (see [2]).

When calibrating the joint default intensities $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ for the CDX.NA.IG Series 9, December 17, 2007 we used 5 groups $I_1, I_2, \ldots, I_5$ where $I_j = \{1, \ldots, i_j\}$ for $i_j = 6, 19, 25, 61, 125$. We label the obligors by decreasing level of riskiness and use the average over 3-year and 5-year CDS spreads as a measure of riskiness. Consequently, obligor 1 has the highest average CDS spread while company 125 has the lowest average CDS spread. Moreover, the obligors in the set $I_5 \setminus I_4$ consisting of the 64 safest companies are assumed to never default individually, and the corresponding CDSs are excluded from the calibration, which in turn relaxes the constraints for $\boldsymbol{\lambda}$ (see [2]). Hence, the obligors in $I_5 \setminus I_4$ can only bankrupt due to a simultaneous default of the companies in the group $I_5 = \{1, \ldots, 125\}$, i.e., in an Armageddon event. With this structure the calibration against the December 17, 2007 data-set is very good as can be seen in Table 1. Note that by resorting to stochastic recoveries, we can get a perfect fit of the same data-set (see [2]).

Table 1 CDX.NA.IG Series 9, December 17, 2007 and iTraxx Europe Series 9, March 31, 2008. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The $[0, 3]$ spread is quoted in %. All maturities are for five years.

CDX 2007-12-17

CDO tranche	$[0,3]$	$[3,7]$	$[7,10]$	$[10,15]$	$[15,30]$
Market spread	48.07	254.0	124.0	61.00	41.00
Model spread	48.07	254.0	124.0	61.00	38.94
Absolute error in bp	0.010	0.000	0.000	0.000	2.061
Relative error in %	0.0001	0.000	0.000	0.000	5.027

iTraxx Europe 2008-03-3

CDO tranche	$[0,3]$	$[3,6]$	$[6,9]$	$[9,12]$	$[12,22]$
Market spread	40.15	479.5	309.5	215.1	109.4
Model spread	41.68	429.7	309.4	215.1	103.7
Absolute error in bp	153.1	49.81	0.0441	0.0331	5.711
Relative error in %	3.812	10.39	0.0142	0.0154	5.218

The calibration of the joint default intensities $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ for the data sampled at March 31, 2008 is more demanding. This time we use 18 groups $I_1, I_2, \ldots, I_{18}$ where $I_j = \left\{1, \ldots, i_j\right\}$ for $i_j = 1, 2, \ldots, 11, 13, 14, 15, 19, 25, 79, 125$. In order to improve the fit, as in the 2007-case, we relax the constraints for $\boldsymbol{\lambda}$ by excluding from the calibration the CDSs corresponding to the obligors in $I_{18} \setminus I_{17}$. Hence, we assume that the obligors in $I_{18} \setminus I_{17}$ never default individually, but can only bankrupt due to an simultaneous default of all companies in the group $I_{18} = \{1, \ldots, 125\}$. In this setting, the calibration of the 2008 data-set with constant recoveries yields an acceptable fit except for the $[3, 6]$ tranche, as can be seen in Table 1. However, by including stochastic recoveries, as illustrated in [2], the fit is substantially improved.

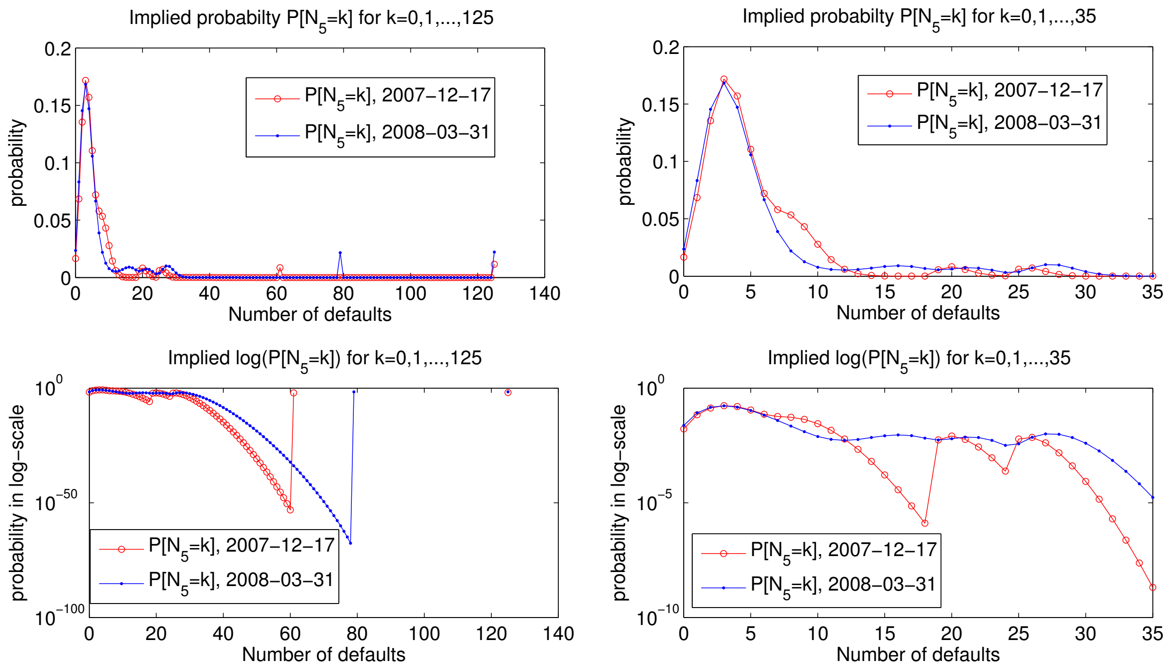

Fig. 2 The implied distribution $\mathbb{P}[N_5 = k]$ on $\{0, 1, \ldots, \ell\}$ where $\ell = 125$ (top left) and $\ell = 35$ (top right) when the model is calibrated against CDX.NA.IG Series 9, December 17, 2007 and iTraxx Europe Series 9, March 31, 2008. The corresponding log distributions $\ln(\mathbb{P}[N_5 = k])$ on $\{0, 1, \ldots, \ell\}$ are displayed bottom left ($\ell = 125$) and bottom right ($\ell = 35$).

After the fit of the model against market spreads we can use the calibrated portfolio parameters $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ together with the calibrated individual default intensities, to study the credit-loss distribution in the portfolio. In this paper we only focus on some few examples derived from the loss distribution with constant recoveries evaluated at $T = 5$ years.

The allowance of joint defaults of the obligors in the groups I_j together with the restriction of the most safest obligors not being able to default individually, will lead to some interesting effects of the loss distribution, as can be seen in Figure 2. For example, we clearly see that the support of the loss-distributions will in practice be limited to a rather compact set. To be more specific, the upper graphs in Figure 2 indicate that $\mathbb{P}[N_5 = k]$ roughly has support on the set $\{1, \ldots, 35\} \cup \{61\} \cup \{125\}$ for the 2007 case and on $\{1, \ldots, 40\} \cup \{79\} \cup \{125\}$ for the 2008 data-set. This becomes even more clear in a log-loss distribution, as is seen in the lower graphs in Figure 2. From the bottom graphs in Figure 2 we see that the default-distribution is nonzero on $\{36, \ldots, 61\}$ in the 2007-case and nonzero on $\{41, \ldots, 79\}$ for the 2008-sample, but the actual size of the loss-probabilities are in the range 10^{-10} to 10^{-70}. Such low values will obviously be treated as zero in any practically relevant computation. Furthermore, the reasons for the empty gap in the bottom left graph in Figure 2 on the interval $\{62, \ldots, 124\}$ for the 2007-case is due to the fact that we forced the obligors in the set $I_5 \setminus I_4$ to never default individually, but only due to an simultaneous common shock default of the companies in the group $I_5 = \{1, \ldots, 125\}$. This Armageddon event is displayed as an isolated nonzero 'dot' at default nr 125 in the lower left graph of Figure 2. The gap on $\{80, \ldots, 124\}$ in the 2008 case is explained similarly due to our assumption on the companies in the set $I_{19} \setminus I_{18}$. Also note that the two 'dots' at default nr 125 in the lower left plot of are manifested as spikes in the upper left graph displayed in Figure 2. The shape of the multimodal loss distributions presented in Figure 2 are typical for models allowing simultaneous defaults, see for example Figure 2, page 59 in [9] and Figure 2, page 710 in [8].

We then use the CDX.NA.IG fitted model of December 17, 2007 to compute implied min-variance hedging strategies for CDO tranches when a portfolio of CDSs is used as hedging instrument. By min-variance we mean a hedge that minimizes the variance of the hedging error (risk-neutral variance relatively to the pricing measure, for tractability reasons). The aim is to analyze the behavior of hedging strategies when the riskiest names of the index are used for hedging standard CDO tranches.

Figure 3 displays the nominal exposure for the 3 (resp. 4, 5 and 6) most riskiest CDSs when hedging one unit of nominal exposure in a CDO tranche. The hedging exposure are derived from the min-variance hedging strategy computed in [1] (see [2] for additional numerical experiments).

Furthermore, Table 2 displays the names and sizes of the 3-year CDS spreads used in the hedging strategy. Each plot in Figure 3 should be interpreted as follows: in every pair (x, y) the x-component represents the size of the 3-year CDS spread at the hedging time $t = 0$ while the y-component is the corresponding nominal CDS-exposure required for hedging. The graphs are ordered from top left to bottom right, where the top panel corresponds to hedging with the $d = 3$ riskiest CDS and the bottom panel corresponds to hedging with the $d = 6$ riskiest names. Note that the x-axes are displayed from the riskiest obligor to the safest. Thus, hedge-sizes y for riskier CDSs are aligned to the left in each plot while y-values for safer CDSs are consequently displayed more to the right. In doing this, going from the topleft to the bottom right panel consists in observing the effect of including new safer CDSs from the right part of the graphs. We have connected the pairs (x, y) with lines forming graphs that visualizes possible trends of the min-variance hedging strategies for the d most riskiest CDSs. When the three riskiest names are used for hedging (top panel), we observe that the amount of nominal exposure in hedging instruments decreases with the degree of subordination, i.e., the [0-3%] equity tranche requires more nominal exposure in CDSs to be hedged than the upper tranches. Note moreover that the min-variance hedging portfolio contains more CDSs on names with lower spreads. When lower-spread CDSs are added in the portfolio, the picture remains almost the same for the 3 riskiest names. For the remaining safer names however, the picture depends on the characteristics of the tranche. For the [0-3%] equity tranche, the quantity of the remaining CDSs required for hedging sharply decrease as additional safer names are added. One possible explanation is that adding too many names in the hedging strategy will be useless when hedging the equity tranche. This is intuitively clear since one expects that the most riskiest obligors will default first and consequently reduce the equity tranche substantially, explaining the higher hedge-ratios for riskier names, while it is less likely that the more safer names will default first and thus incur losses on the first tranche which explains the lower hedge ratios for the safer names. We observe the opposite trend for the senior (safer) tranches: adding new (safer) names in the hedging portfolio seems to be useful for "non equity" tranches since the nominal exposure required for these names increases when they are successively added.

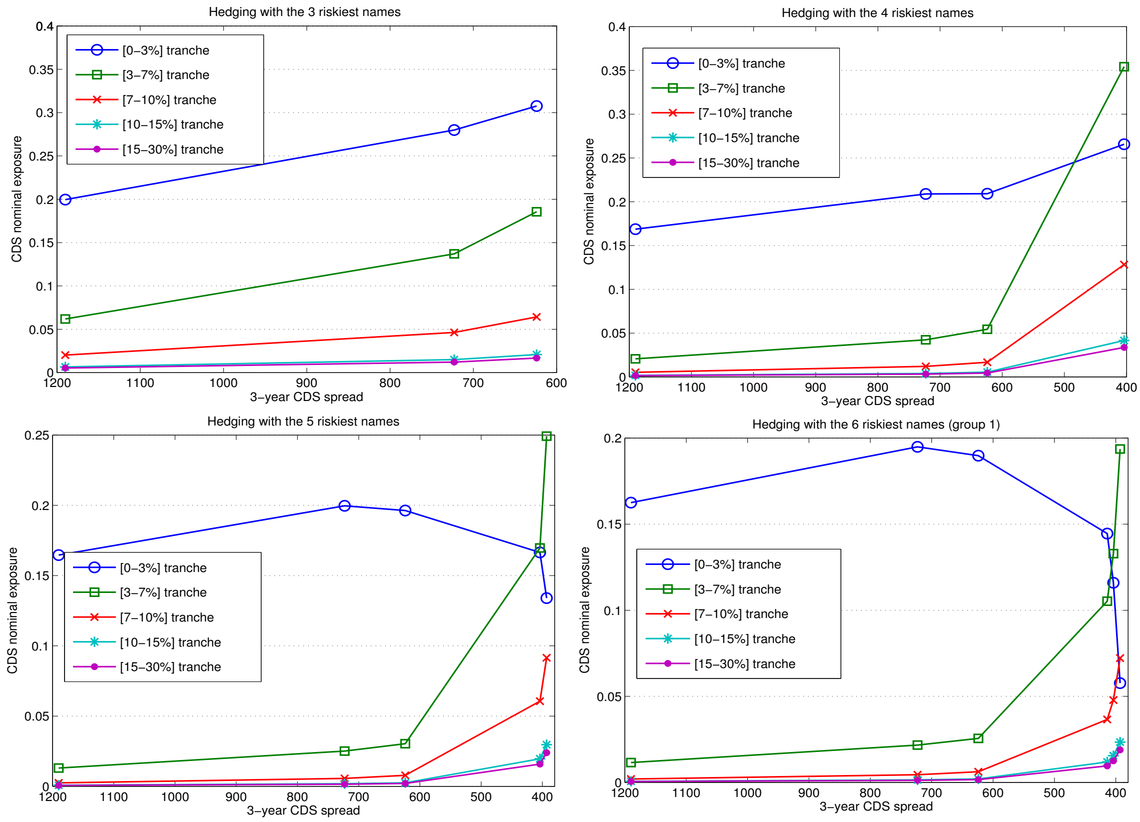

Fig. 3 Min-variance hedging strategies associated with the d riskiest CDS, $d = 3, 4, 5, 6$ for one unit of nominal exposure of different CDO tranches in a model calibrated to market spreads of CDX.NA.IG Series 9 on December 17, 2007.

Table 2 The names and CDS spreads (in bp) of the six riskiest obligors used in the hedging strategy displayed by Figure 3.

Company (Ticker)	CCR-HomeLoans	RDN	LEN	SFI	PHM	CTX
3-year CDS spread	1190	723	624	414	404	393

4 Conclusions

We construct a bottom-up dynamic model of portfolio credit risk in which pricing, hedging and counterparty risk valuation can be made in a theoretical sound and practical convenient way. This boils down to two possible equivalent perspectives of the model. The Markov copula perspective (see [1]) allows us to derive min-variance hedging strategies for portfolio credit derivatives and to propose a two-step efficient calibration procedure of individual default marginals and dependence structure. The common-shock perspective (see [2]) underlies semi-explicit convolution-based pricing schemes to assess the credit portfolio loss distribution at several time horizons. Then, calibration of model parameters on both single-name and portfolio credit derivative products, hedging or CVA computations are numerically tractable in this setting.

Appendix

Here is the proof of Proposition 2.1. For $I \subseteq N$, we define the filtration $\mathbb{X}^I = (\mathcal{X}^I_t)_{t\geq 0}$ as the initial enlargement of $\mathbb{X}$ by the τ_i for $i \in I$, i.e. for every t :

$$\mathcal{X}^I_t = \mathcal{X}_t \vee \bigvee_{i\in I} \sigma(\tau_i).$$

By an application of Lemma 2.5 in [14], writing $J = N \setminus I$ for every $I \subseteq N$, we obtain:

$$\mathbb{P}\left(\tau_1 > t_1, \ldots, \tau_n > t_n \mid \mathcal{F}_t\right) = \sum_{I \subsetneq N} \mathbb{1}_{\{I_t = I\}} \mathbb{1}_{\{\tau_i > t_i,\, i\in I\}} \frac{\mathbb{P}(\tau_j > t \vee t_j,\, j \in J \,|\, \mathcal{X}^I_t)}{\mathbb{P}(\tau_j > t,\, j \in J \,|\, \mathcal{X}^I_t)}. \tag{8}$$

Now, in the common shocks model of this paper, writing

$$\mathcal{Y}_J = \{Y \in \mathcal{Y};\, Y \cap J \neq \emptyset\},\ \ \bar{\mathcal{Y}}_J = \mathcal{Y} \setminus \mathcal{Y}_J$$

$$\tau^J_i = \min_{\{Y \in \bar{\mathcal{Y}}_J;\, i\in Y\}} \tau_Y,\ \ \bar{\mathcal{X}}^I_t = \mathcal{X}_t \vee \bigvee_{i\in I} \sigma(\tau^J_i)$$

$$\bar{t}_Y = \max_{j \in Y\cap J} (t \vee t_j),\ \ \bar{t} = \max_{Y\in\mathcal{Y}} \bar{t}_Y = \max_{Y\in\mathcal{Y}_J} \bar{t}_Y,$$

we have on $\{I_t = I\}$ (and therefore $\{\tau_i = \tau^J_i,\, i \in I\}$):

$$\begin{aligned}
\mathbb{P}(\tau_j > & t \vee t_j,\, j \in J \,|\, \mathcal{X}^I_t) = \mathbb{P}(\tau_j > t \vee t_j,\, j \in J \,|\, \bar{\mathcal{X}}^I_t) \\
&= \mathbb{P}(\tau_Y > \bar{t}_Y,\, Y \in \mathcal{Y}_J \,|\, \bar{\mathcal{X}}^I_t) = \mathbb{P}\left(E_Y > \Lambda_{\bar{t}_Y},\, Y \in \mathcal{Y}_J \,|\, \bar{\mathcal{X}}^I_t\right) \\
&= \mathbb{E}\left(\mathbb{P}\left(E_Y > \Lambda_{\bar{t}_Y},\, Y \in \mathcal{Y}_J \,|\, \bar{\mathcal{X}}^I_{\bar{t}}\right) \,|\, \bar{\mathcal{X}}^I_t\right) = \mathbb{E}\left\{\exp\left(-\sum_{Y\in\mathcal{Y}_J} \Lambda^Y_{\bar{t}_Y}\right) \Big|\, \bar{\mathcal{X}}^I_t\right\}
\end{aligned}$$

where $\mathbb{P}\left(E_Y > \Lambda_{\bar{t}_Y},\, Y \in \mathcal{Y}_J \,|\, \bar{\mathcal{X}}^I_{\bar{t}}\right) = \exp\left(-\sum_{Y\in\mathcal{Y}_J} \Lambda^Y_{\bar{t}_Y}\right)$ in the last identity holds by independence of $(E_Y)_{Y\in\mathcal{Y}_J}$ from $\bar{\mathcal{X}}^I_{\bar{t}}$ and by $\bar{\mathcal{X}}^I_{\bar{t}}$- (in fact, $\mathcal{X}_{\bar{t}}$-) measurability of $(\Lambda^Y_{\bar{t}_Y})_{Y\in\mathcal{Y}_J}$.

Note moreover that we have on $\{I_t = I\}$:

$$\mathcal{X}_t \subseteq \bar{\mathcal{X}}^I_t \subseteq \mathcal{X}_t \vee \bigvee_{Y\in\bar{\mathcal{Y}}_J} \sigma(E_Y),$$

where the $E_Y, Y \in \bar{\mathcal{Y}}_J$ are independent from $\mathbb{X}$. We thus have on $\{I_t = I\}$:

$$\mathbb{E}\left\{\exp\left(-\sum_{Y\in\mathcal{Y}_J} \Lambda^Y_{\bar{t}_Y}\right) \Big| \mathcal{X}_t\right\} = \mathbb{E}\left\{\exp\left(-\sum_{Y\in\mathcal{Y}_J} \Lambda^Y_{\bar{t}_Y}\right) \Big| \mathcal{X}_t \vee \bigvee_{Y\in\bar{\mathcal{Y}}_J} \sigma(E_Y)\right\}$$

$$= \mathbb{E}\left\{\exp\left(-\sum_{Y\in\mathcal{Y}_J} \Lambda^Y_{\bar{t}_Y}\right) \Big| \bar{\mathcal{X}}^I_t\right\} = \mathbb{P}\left(\tau_j > t \vee t_j,\ j \in J \,|\, \mathcal{X}^I_t\right).$$

The Markov property of $\mathbf{X}_t$ finally yields that

$$\mathbb{P}(\tau_j > t \vee t_j, j \in J \,|\, \mathcal{X}^I_t) = \exp\left(-\sum_{Y\in\mathcal{Y}_J} \Lambda^Y_t\right) \mathbb{E}\left\{\exp\left(-\sum_{Y\in\mathcal{Y}_J} \Lambda^Y_{t,\bar{t}_Y}\right) \Big| \mathbf{X}_t\right\}$$

$$= \mathbb{P}(\tau_j > t,\ j \in J \,|\, \mathcal{X}^I_t)\mathbb{E}\left\{\exp\left(-\sum_{Y\in\mathcal{Y}_J} \Lambda^Y_{t,\bar{t}_Y}\right) \Big| \mathbf{X}_t\right\}.$$

Plugging this into (8) yields that

$$\mathbb{P}\left(\tau_1 > t_1, \ldots, \tau_n > t_n \mid \mathcal{F}_t\right) = \sum_{I\subsetneq N} \mathbb{1}_{\{I_t = I\}} \mathbb{1}_{\{\tau_i > t_i,\ i\in I\}} \mathbb{E}\left\{\exp\left(-\sum_{Y\in\mathcal{Y}} \Lambda^Y_{t,\bar{t}_Y}\right) \Big| \mathbf{X}_t\right\},$$

which is (3). Setting all t_j but t_i equal to 0, we deduce (4) from (3), for $t_i \geq t$.

Acknowledgements

The research of T.R. Bielecki was supported by NSF Grant DMS–0604789 and NSF Grant DMS–0908099. The research of A. Cousin benefited from the support of the DGE and the ANR project Ast&Risk. The research of S. Crépey benefited from the support of the "Chaire Risque de Crédit" and of the "Chaire Marchés en Mutation", Fédération Bancaire Française. The research of A. Herbertsson was supported by the Jan Wallander and Tom Hedelius Foundation and by Vinnova.

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