A Bottom-Up Dynamic Model of Portfolio Credit Risk. Part II: Common-Shock Interpretation, Calibration and Hedging issues

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Abstract

In this paper, we prove that the conditional dependence structure of default times in the Markov model of [4] belongs to the class of Marshall-Olkin copulas. This allows us to derive a factor representation in terms of "common-shocks", the latter beeing able to trigger simultaneous defaults in some pre-specified groups of obligors. This representation depends on the current default state of the credit portfolio so that fast convolution pricing schemes can be exploited for pricing and hedging credit portfolio derivatives. As emphasized in [4], the innovative breakthrough of this dynamic bottom-up model is a suitable decoupling property between the dependence structure and the default marginals as in [10] (like in static copula models but here in a full-flesh dynamic "Markov copula" model). Given the fast deterministic pricing schemes of the present paper, the model can then be jointly calibrated to single-name and portfolio data in two steps, as opposed to a global joint optimization procedures involving all the model parameters at the same time which would be untractable numerically. We illustrate this numerically by results of calibration against market data from CDO tranches as well as individual CDS spreads. We also discuss hedging sensitivities computed in the models thus calibrated.

Keywords: Portfolio credit risk, Basket credit derivatives, Markov copula model, Common shocks, Pricing, Calibration, Min-variance hedging.

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1 Introduction

In [4] we introduced a Markov copula model of default times providing a decoupling between the single-name marginals and the dependence structure of the default times. In this sense, this model solves the portfolio credit risk top-down bottom-up puzzle [8]. For earlier partial progress in this direction, see [11, 12, 14] and the discussion in [4]. This paper is the applicative companion to [4] (see also [3] for a short announcing version of both papers and [5] for further features of the model).

The paper is organized as follows. In Section 2 we provide an "executive summary" of the dynamic model of [4] in the form of an equivalent common-shock representation. In terms of common-shock representation, we mean that each individual default process can be represented by Cox processes likely to trigger defaults simultaneously in some prespecifed group of obligors. In Section 3 we use the resulting representation to derive fast deterministic pricing algorithms. As emphasized in [4], the innovative breakthrough of the model is a suitable decoupling property between the dependence structure and the individual names [10], like in static copula models but here in a full-flesh dynamic "Markov copula" model. Given the fast deterministic pricing schemes of this paper, the model can then be jointly calibrated to single-name and portfolio data in two steps, as opposed to a global joint optimization procedures involving all the model parameters at the same time which would be untractable numerically. In particular, this model allows one to address in a dynamic and theoretically consistent way the issues of hedging basket credit derivatives by individual names, whilst preserving the static common factor tractability. To illustrate these features, Section 4 presents numerical results of calibration against market data from CDO tranches as well as individual CDS spreads and Section 5 discusses hedging sensitivities computed in the models thus calibrated. Note that this model can be applied well the space of consistent valuation and hedging of CDSs and CDOs. In particular it is used in [2, 9, 13] (see also [6]) for valuation and hedging of counterparty risk on credit derivatives.

In the rest of the paper we consider a risk neutral pricing model $(\Omega, \mathcal{F}, \mathbb{P})$, for a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ which will be specified below and where $T \geq 0$ is a fixed time horizon. We denote $N_n = \{1, \dots, n\}$ and let $\mathcal{N}_n$ denote the set of all subsets of N_n where n represents the number of obligors in the underlying credit portfolio.

2 Model of Default Times

In the Markov copula common shocks model, defaults are the consequence of some "shocks" associated with groups of obligors. We define the following pre-specified set of groups

$$\mathcal{Y} = \{\{1\}, \dots, \{n\}, I_1, \dots, I_m\},$$

where $I_1, \dots, I_m$ are subsets of $\{1, \dots, n\}$, and each group I_j contains at least two obligors or more. The shocks are divided in two categories: the "idiosyncratic" shocks associated with singletons $\{1\}, \dots, \{n\}$ can only trigger the default of name $1, \dots, n$ individually, while the "systemic" shocks associated with multi-name groups $I_1, \dots, I_m$ may simultaneously trigger the default of all names in these groups. Note that several groups I_j may contain a given name i, so that only the shock occurring first effectively triggers the default of that name. As a result, when a shock associated with a specific group occurs at time t, it only triggers the default of names that are still alive in that group at time t. In the following, the elements Y of $\mathcal{Y}$ will be used to designate shocks and we let $\mathcal{I} = (I_l)_{1 \le l \le m}$ denote the pre-specified set of multi-name groups of obligors. Shock intensities $\lambda_Y(t, \mathbf{X}_t)$ will be specified later in terms of a Markovian factor process $\mathbf{X}_t$. Letting $\Lambda_t^Y = \int_0^t \lambda_Y(s, \mathbf{X}_s) ds$, we define

$$\widehat{\tau}_Y = \inf\{t > 0;\, \Lambda_t^Y > E_Y\}, \tag{1}$$

where the random variables E_Y are i.i.d. and exponentially distributed with parameter 1. For every obligor i we let

$$\tau_i = \min_{\{Y \in \mathcal{Y};\, i \in Y\}} \widehat{\tau}_Y, \tag{2}$$

which defines the default time of obligor i in the common shocks model. The model filtration is given as $\mathbb{F} = \mathbb{X} \vee \mathbb{H}$, the filtration generated by the factor process $\mathbf{X}$ and the point process $\mathbf{H} = (H^i)_{1 \le i \le n}$ with $H_t^i = \mathbb{1}_{\tau_i \le t}$.

This model can be viewed as a doubly stochastic (via the stochastic intensities Λ^Y) and dynamized (via the introduction of the filtration $\mathbb{F}$) generalization of the Marshall-Olkin model [18]. The purpose of the factor process $\mathbf{X}$ is to more realistically model diffusive randomness of credit spreads. Note that in [4], we present the model the reverse way round, i.e. we first construct a suitable Markov process $(\mathbf{X}_t, \mathbf{H}_t)$ and then define the τ_i as the jump times of the H^i.

The set of obligors alive (resp. in default) at time t is denoted by $J_t = \mathrm{supp}^c(\mathbf{H}_t)$ (resp. $H_t = \mathrm{supp}(\mathbf{H}_t)$). For every $Y \in \mathcal{Y}$ and every set of nonnegative constants $t, t_1, \ldots, t_n$, we define

$$\theta_t^Y = \max_{i \in Y \cap J_t} t_i$$

(with the convention that $\max \emptyset = -\infty$). Note that $Y \cap J_t$ in θ_t^Y represents the set of survivors in Y at time t. We also write

$$\Lambda_{s,t}^Y = \int_s^t \lambda_Y(s, X_s^Y) ds, \;\; \lambda_t^i = \sum_{\{Y \in \mathcal{Y};\, i \in Y\}} \lambda_Y(t, \mathbf{X}_t). \tag{3}$$

The following result (see [3, 4]) is key in the model.

Proposition 2.1 *For any fixed nonnegative constants $t, t_1, \ldots, t_n$, we have:*

$$\mathbb{P}\left(\tau_1 > t_1, \ldots, \tau_n > t_n \mid \mathcal{F}_t\right) = \mathbb{1}_{\{t_i < \tau_i\,,\ i \in I_t\}} \mathbb{E}\left\{ \exp\big(- \sum_{Y \in \mathcal{Y}} \Lambda^Y_{t, t \vee \theta_t^Y} \big) \,\Big|\, \mathbf{X}_t \right\}. \tag{4}$$

In particular, for every obligor i and $t_i \ge t$,

$$\mathbb{P}(\tau_i > t_i \,|\, \mathcal{F}_t) = \mathbb{1}_{\{\tau_i > t\}} \mathbb{E}\left\{ \exp\big(- \int_t^{t_i} \lambda_s^i ds \big) \,|\, \mathbf{X}_t \right\}. \tag{5}$$

In Section 4, we will see that thanks to formula (4) (resp. (5) and under an additional affine structure postulated below on each individual pre-default intensity process λ_t^i), efficient convolution recursion procedures (resp. affine methodologies) are available for pricing multi-name credit derivatives like CDO tranches (resp. single-name credit derivatives like CDSs), conditionally on any given state of the dynamic model $(\mathbf{X}_t, \mathbf{H}_t)$. The model can then be calibrated in two steps: individual λ^i-parameters are first calibrated to individual CDSs and the model dependence λ_I-parameters are then calibrated to CDO tranches (as opposed to a global joint optimization procedures involving all the model parameters at the same time, which would be untractable numerically). But first, as announced above, in order to ensure the Markov consistency and Markov copula feature of the setup (see [10]), we assume further that every individual process λ^i is an affine process (in particular, a Markov process), as in either specification below. Consequently the conditioning with respect to $\mathbf{X}_t$ can be replaced by a conditioning with respect to λ^i_t in (5), hence exponential-affine methodogies for computing (5) follow.

Example 2.2 **(i) (Deterministic group intensities)**. The idiosyncratic intensities $\lambda_{\{i\}}(t, \mathbf{X}_t)$ are affine, and the systemic intensities $\lambda_Y(t, \mathbf{X}_t)$ are deterministic functions of time, i.e. the functions $\lambda_Y(t, \mathbf{x})$ do not depend on $\mathbf{x}$, for $Y \in \mathcal{Y}$ that are not singletons.
(ii) (Extended CIR intensities). $\mathbf{X}_t = (X^Y_t)_{Y \in \mathcal{Y}}$ and for every $Y \in \mathcal{Y}$, $\lambda_Y(t, \mathbf{X}_t) = X^Y_t$, where X^Y_t is an extended CIR process

$$dX^Y_t = a(b_Y(t) - X^Y_t)dt + c\sqrt{X^Y_t}dW^Y_t, \tag{6}$$

for nonnegative constants a, c (independent of Y) and a nonnegative function $b_Y(t)$, and where the W^Y are independent standard Brownian motions.

In the second specification, affinity of λ^i (which is trivial in the first specification) arises from the fact that the SDE for the factors X^Y have the same coefficients except for the $b_Y(t)$. Thus, $X^i := \sum_{\{Y \in \mathcal{Y};\, i \in Y\}} X^Y$ satisfies the following extended CIR SDE:

$$dX^i_t = a(b_i(t) - X^i_t)dt + c\sqrt{X^i_t}dW^i_t, \tag{7}$$

for the function $b_i(t) = \sum_{\{Y \in \mathcal{Y};\, i \in Y\}} b_Y(t)$ and the Brownian motion

$$dW^i_t = \sum_{i \in Y} \frac{\sqrt{X^Y_t}}{\sqrt{\sum_{i \in Y} X^Y_t}} dW^Y_t.$$

3 Fast Deterministic Pricing Schemes

3.1 Common Shocks Model Interpretation

In view of formula (4), conditionally on any given state $(\mathbf{x}, \mathbf{k})$ of $(\mathbf{X}, \mathbf{H})$ at time t, it is possible to define a "conditional common shock model" of default times of the surviving names at time t, such that the law of the default times in the conditional common shock model is the same as the corresponding conditional distribution in the original model. This representation will be used in the next section for deriving fast exact convolution recursion procedures for pricing portfolio loss derivatives.

We thus introduce a family of common shocks copula models, parameterized by the current time t. For every $Y \in \mathcal{Y}$, we define

$$\tau_Y(t) = \inf\{\theta > t;\, \Lambda^Y_\theta > \Lambda^Y_t + E_Y\},$$

where the random variables E_Y are i.i.d. and exponentially distributed with parameter 1. For every obligor i we let

$$\tau_i(t) = \min_{\{Y \in \mathcal{Y};\, i \in Y\}} \tau_Y(t) \;, \tag{8}$$

which defines the default time of obligor i in the common shocks copula model starting at time t. We also introduce the indicator processes $H^Y_\theta(t) = \mathbb{1}_{\{\tau_Y(t) \le \theta\}}$ and $H^i_\theta(t) = \mathbb{1}_{\{\tau_i(t) \le \theta\}}$, for every shock Y, obligor i and time horizon $\theta \ge t$. Let $Z \in \mathcal{N}_n$ denote a set of obligors, meant in the probabilistic interpretation to represent the set J_t of survived obligors in the original model at time t. We now prove that on $\{J_t = Z\}$, the conditional law of $(\tau_i)_{i \in J}$ given $\mathcal{F}_t$ in the original model, is equal to the conditional law of $(\tau_i(t))_{i \in Z}$ given $\mathbf{X}_t$ in the common shocks framework starting at time t. Let also $N_\theta = \sum_{1 \le i \le n} H^i_\theta$ denote the cumulative number of defaulted obligors in the original model up to time θ. Let $N_\theta(t, Z) = n - |Z| + \sum_{i \in Z} H^i_\theta(t)$, denote the cumulative number of defaulted obligors in the time-t common shocks framework up to time θ where $|Z|$ is the cardinality of the set Z.

Proposition 3.1 *Let $Z \in \mathcal{N}_n$ denote an arbitrary subset of obligors and let $t \ge 0$. Then,*
(i) *for every $t_1, \ldots, t_n \ge t$, one has*

$$\mathbb{1}_{\{J_t = Z\}} \mathbb{P}\left(\tau_i > t_i,\, i \in J_t \,\middle|\, \mathcal{F}_t\right) = \mathbb{1}_{\{J_t = Z\}} \mathbb{P}\left(\tau_i(t) > t_i,\, i \in Z \,\middle|\, \mathbf{X}_t\right). \tag{9}$$

(ii) *for every $\theta \ge t$, one has that for every $k = n - |Z|, \ldots, n$,*

$$\mathbb{1}_{\{J_t = Z\}} \mathbb{P}\left(N_\theta = k \,\middle|\, \mathcal{F}_t\right) = \mathbb{1}_{\{J_t = Z\}} \mathbb{P}\left(N_\theta(t, Z) = k \,\middle|\, \mathbf{X}_t\right).$$

Proof. Part (ii) readily follows from part (i), that we now show. Let, for every obligor i, $\tilde{t}_i = \mathbb{1}_{i \in J_t} t_i$. Note that one has, for $Y \in \mathcal{Y}$

$$\max_{i \in Y \cap J_t} \tilde{t}_i = \max_{i \in Y \cap J_t} t_i = \theta^Y_t.$$

Thus, by application of formula (4) to the sequence of times $(\tilde{t}_i)_{1 \le i \le n}$, it comes,

$$\begin{aligned}
&\mathbb{1}_{\{J_t = Z\}} \mathbb{P}\left(\tau_i > t_i,\, i \in J_t \,\middle|\, \mathcal{F}_t\right) \\
&= \mathbb{1}_{\{J_t = Z\}} \mathbb{P}\left(\left(\tau_i > t_i,\, i \in Z\right),\, \left(\tau_i > 0,\, i \in Z^c\right) \,\middle|\, \mathcal{F}_t\right) \\
&= \mathbb{1}_{\{J_t = Z\}} \mathbb{E}\left\{\exp\big(-\sum_{Y \in \mathcal{Y}} \Lambda^Y_{t, \theta^Y_t}\big) \,\middle|\, \mathbf{X}_t\right\}
\end{aligned}$$

which on $\{J_t = Z\}$ coincides with the expression

$$\mathbb{E}\left\{\exp\big(-\sum_{Y \in \mathcal{Y}} \Lambda^Y_{t, \max_{i \in Y \cap Z} t_i}\big) \,\middle|\, \mathbf{X}_t\right\}$$

deduced from (4) for $\mathbb{P}(\tau_i(t) > t_i,\, i \in Z \,|\, \mathbf{X}_t)$. □

3.2 Recursive Convolutive Pricing Schemes

In this subsection we use the conditional common shock model representation to derive fast convolution recursion algorithms for computing the conditional portfolio loss distribution. In the case where the recovery rate is the same for all names, i.e., $R_i = R$, $i = 1, \dots, n$, the price process for a CDO tranche $[a, b]$ is determined by the probabilities $\mathbb{P}\,[N_\theta = k \,|\, \mathcal{F}_t]$ for $k = |\mathbf{H}_t|, \dots, n$ and $\theta \geq t \geq 0$. But we know from Proposition 3.1(ii) that

$$\mathbb{P}\,[N_\theta = k \,|\, \mathcal{F}_t] = \mathbb{P}\,[N_\theta(t, Z) = k \,|\, \mathbf{X}_t]$$

on the event $\{J_t = Z\}$, so we will focus on computation of the latter probabilities, which are derived in formula (11) below.

We henceforth assume a nested structure of the sets I_j given by

$$I_1 \subset \ldots \subset I_m. \tag{10}$$

This structure implies that if all obligors in group I_k have defaulted, then all obligors in group $I_1, \dots, I_{k-1}$ have also defaulted. As we shall detail in Remark 3.3, the nested structure (10) yields a particularly tractable expression for the portfolio loss distribution. This nested structure also makes sense financially with regards to the hierarchical structure of risks which is reflected in standard CDO tranches.

Remark 3.2 A dynamic group structure would be preferable from a financial point of view. In the same vein one could deplore the absence of direct contagion effects in this model (it only has joint defaults). However it should be stressed that we are building a pricing model, not an econometric model; the applications we have in mind are hedging CDO tranches by individual names (see Section 5), as well as valuation and hedging of counterparty risk on credit portfolios (see [7]). In these regards, efficient pricing (at any future point in time, not only at time 0 [7]) and Greeking procedures, as well as efficient joint calibration to CDS and CDO data (see Section 4), are the main issues, and these are already quite difficult to achieve simultaneously in a single model.

Denoting conventionally $I_0 = \emptyset$ and $H_\theta^{I_0}(t) = 1$, then in view of (10), the events

$$\Omega_\theta^j(t) := \{H_\theta^{I_j}(t) = 1, H_\theta^{I_{j+1}}(t) = 0, \dots, H_\theta^{I_m}(t) = 0\}, \;\; 0 \leq j \leq m$$

form a partition of Ω. Hence, we have

$$\mathbb{P}(N_\theta(t, Z) = k \,|\, \mathbf{X}_t) = \sum_{0 \leq j \leq m} \mathbb{P}\big(N_\theta(t, Z) = k \,|\, \Omega_\theta^j(t), \mathbf{X}_t\big)\mathbb{P}\big(\Omega_\theta^j(t) \,|\, \mathbf{X}_t\big) \tag{11}$$

where, by construction of the $H_\theta^I(t)$ and independence of the $\lambda_I(t, X_t^I)$ we have

$$\mathbb{P}\big(\Omega_\theta^j(t) \,|\, \mathbf{X}_t\big) = \Big(1 - \mathbb{E}\big(e^{-\Lambda_{t,\theta}^{I_j}} \,|\, X_t^{I_j}\big)\Big) \prod_{j+1 \leq l \leq m} \mathbb{E}\big(e^{-\Lambda_{t,\theta}^{I_l}} \,|\, X_t^{I_l}\big) \tag{12}$$

which in our model can be computed very quickly (actually, semi-explicitly in either of the specifications of Example (2.2)). We now turn to the computation of the term

$$\mathbb{P}\big(N_\theta(t, Z) = k \,|\, \Omega_\theta^j(t), \mathbf{X}_t\big) \tag{13}$$

appearing in (11). Recall first that $N_\theta(t,Z) = n - |Z| + \sum_{i\in Z} H^i_\theta(t)$ with $|Z|$ denoting the cardinality of Z. We know that for every group $j = 1,\dots,m$, given $\Omega^j_\theta(t)$, the marginal default indicators $H^i_\theta(t)$ for $i \in Z$ are such that:

$$H^i_\theta(t) = \begin{cases} 1, & i \in I_j, \\ H^{\{i\}}_\theta(t), & \text{else.} \end{cases} \tag{14}$$

Hence, the $H^i_\theta(t)$ are conditionally independent given $\Omega^j_\theta(t)$. Finally, conditionally on $(\Omega^j_\theta(t), \mathbf{X}_t)$ the random vector $\mathbf{H}_\theta(t) = (H^i_\theta(t))_{i\in N_n}$ is a vector of independent Bernoulli random variables with parameter $p = (p^{i,j}_\theta(t))_{i\in N_n}$, where

$$p^{i,j}_\theta(t) = \begin{cases} 1, & i \in I_j, \\ 1 - \mathbb{E}\left\{\exp\left(-\Lambda^{\{i\}}_{t,\theta}\right) \mid X^{\{i\}}_t\right\}, & \text{else} \end{cases} \tag{15}$$

The conditional probability (13) can therefore be computed by a standard convolution recursive procedure (see, for instance, Andersen and Sidenius [1]).

Remark 3.3 The linear number of terms in the sum of (11) is due to the nested structure of the groups I_j in (11). Note that a convolution recursion procedure is possible for an arbitrary structuring of the groups I_j. However, a general structuring of the m groups I_j would imply 2^m terms instead of m in the sum of (11), which in practice would only work for very few groups m. The nested structure (11) of the I_j, or equivalently, the tranched structure of the $I_j \setminus I_{j-1}$, is also quite natural from the point of view of application to CDO tranches.

4 Model Calibration

In this section we briefly discuss the calibration of the model and some few numerical results connected to the loss-distributions. Subsection 4.1 outlines the calibration methodology with piecewise constant default intensities and constant recoveries (for a calibration with stochastic intensities and/or random recoveries see in [5]). Then Subsection 4.2 presents the numerical calibration of the Markov copula model against market data. We also study the implied loss-distributions in our fitted model for the case with constant recoveries.

4.1 Calibration Methodology

In this subsection we discuss one of the calibration methodologies that will be used when fitting the Markov copula model against CDO tranches on the iTraxx Europe and CDX.NA.IG series in Subsection 4.2. This first calibration methodology will use piecewise constant default intensities and constant recoveries in the convolution pricing algorithm of Subsection 3.

The first step is to calibrate the single-name CDS for every obligor. Given the T-year market CDS spread S^*_i for obligor we want to find the individual default parameters for obligor i so that $P^i_0(S^*_i) = 0$, so

$$S^*_i = \frac{(1-R_i)\mathbb{P}(\tau_i < T)}{\mathrm{h}\sum_{0<t_j\le T} \mathbb{P}(\tau_i > t_j)} \tag{16}$$

where we used the facts that interest rate is zero and that the recovery R_i is constant. Hence, the first step is to extract the implied hazard function $\Gamma_i^*(t) = -\ln \mathbb{P}(\tau_i > T)$ from the CDS curve of every obligor i by using a standard bootstrapping procedure based on (16).

Given the marginal hazard functions, the law of the total number of defaults at a fixed horizon is a function of the joint default intensity functions $\lambda_I(t)$, as described by the recursive algorithm of Subsection 3. The second step is therefore to calibrate the common-shock intensities $\lambda_I(t)$ so that the model CDO tranche spreads coincide with the corresponding market spreads. This is done by using the recursive algorithm of Subsection 3, for $\lambda_I(t)$s parameterized as non-negative and piecewise constant functions of time. Moreover, in view of the definition of λ_t^i in (3), for every obligor i and at each time t we impose the constraint

$$\sum_{I \in \mathcal{I};\, i \in I} \lambda_I(t) \le \lambda_i^*(t) \tag{17}$$

where $\lambda_i^* := \frac{d\Gamma_i^*}{dt}$ denotes the hazard rate (or hazard intensity) of name i. For constant joint default intensities $\lambda_I(t) = \lambda_I$ the constraints (17) reduce to

$$\sum_{I \ni i} \lambda_I \le \underline{\lambda}_i := \inf_{t \in [0,T]} \lambda_i^*(t) \quad \text{for every obligor } i.$$

Given the nested structure of the groups I_j-s specified in (10), this is equivalent to

$$\sum_{j=l}^{m} \lambda_{I_j} \le \underline{\lambda}_{I_l} := \min_{i \in I_l \setminus I_{l-1}} \underline{\lambda}_i \text{ for every group } l. \tag{18}$$

Furthermore, for piecewise constant common shock intensities on a time grid (T_k), the condition (18) extends to the following constraint

$$\sum_{j=l}^{m} \lambda_{I_j}^k \le \underline{\lambda}_{I_l}^k := \min_{i \in I_l \setminus I_{l-1}} \underline{\lambda}_i^k \quad \text{for every } l, k \quad \text{where } \underline{\lambda}_i^k := \inf_{t \in [T_{k-1}, T_k]} \lambda_i^*(t). \tag{19}$$

We remark that insisting on calibrating all CDS names in the portfolio, including the safest ones, implies via (18) or (19) a very constrained region for the common shock parameters. This region can be expanded by relaxing the system of constraints for the joint default intensities, by excluding the safest CDSs from the calibration.

In this paper we will use a time grid consisting of two maturities T_1 and T_2. Hence, the single-name CDSs constituting the entities in the credit portfolio are bootstrapped from their market spreads for $T = T_1$ and $T = T_2$. This is done by using piecewise constant individual default intensity λ_i-s on the time intervals $[0, T_1]$ and $[T_1, T_2]$.

Before we leave this subsection, we give some few more details on the calibration of the common shock intensities for the m groups in the second calibration step. From now on we assume that the joint default intensities $\{\lambda_{I_j}(t)\}_{j=1}^m$ are piecewise constant functions of time, so that $\lambda_{I_j}(t) = \lambda_{I_j}^{(1)}$ for $t \in [0, T_1]$ and $\lambda_{I_j}(t) = \lambda_{I_j}^{(2)}$ for $t \in [T_1, T_2]$ and for every group j. Next, the joint default intensities $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k} = \{\lambda_{I_j}^{(k)} : j = 1, \ldots, m \text{ and } k = 1, 2\}$ are then calibrated so that the five-year model spread $S_{a_l, b_l}(\boldsymbol{\lambda}) =: S_l(\boldsymbol{\lambda})$ will coincide with the corresponding market spread S_l^* for each tranche l. To be more specific, the parameters

$\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ are obtained according to

$$\boldsymbol{\lambda} = \underset{\widehat{\boldsymbol{\lambda}}}{\operatorname{argmin}} \sum_{l} \left(\frac{S_l(\widehat{\boldsymbol{\lambda}}) - S_l^*}{S_l^*} \right)^2 \tag{20}$$

under the constraints that all elements in $\boldsymbol{\lambda}$ are nonnegative and that $\boldsymbol{\lambda}$ satisfies the inequalities (19) for every group I_l and in each time interval $[T_{k-1}, T_k]$ where $T_0 = 0$. In $S_l(\widehat{\boldsymbol{\lambda}})$ we have emphasized that the model spread for tranche l is a function of $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ but we suppressed the dependence in other parameters like interest rate, payment frequency or λ_i, $i = 1, \ldots, n$.

4.2 Calibration Results

In all the numerical calibrations below we use an interest rate of 3%, the payments in the premium leg are quarterly and the integral in the default leg is discretized on a quarterly mesh. The constant recoveries for all obligors are set equal to 40%. We use Matlab in our numerical calculations and the related objective function is minimized under the suitable constraints by using the built in optimization routine `fmincon` (e.g. in the setup of Subsection 4.1, minimizing the criterion (20) under the constraints given by equations on the form (19)).

In this subsection we calibrate our model against CDO tranches on the iTraxx Europe and CDX.NA.IG series with maturity of five years. We use the calibration methodologies described in Subsection 4.1.

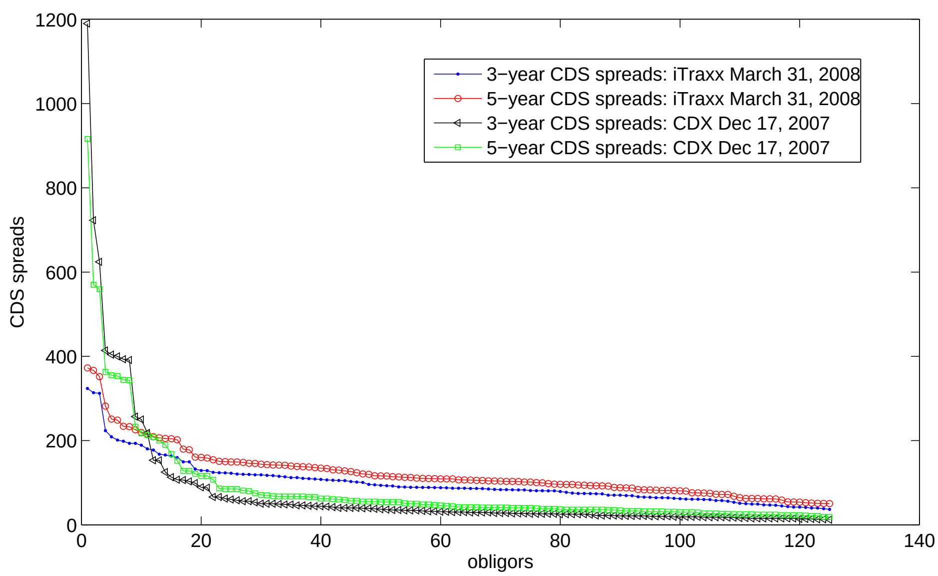


Figure 1: The 3 and 5-year market CDS spreads for the 125 obligors used in the single-name bootstrapping, for the two portfolios CDX.NA.IG sampled on December 17, 2007 and the iTraxx Europe series sampled on March 31, 2008. The CDS spreads are sorted in decreasing order.

Hence, the 125 single-name CDSs constituting the entities in these series are bootstrapped from their market spreads for $T_1 = 3$ and $T_2 = 5$ using piecewise constant individual default intensities on the time intervals $[0, 3]$ and $[3, 5]$. Figure 1 displays the 3 and 5-year market CDS spreads for the 125 obligors used in the single-name bootstrapping, for the two portfolios CDX.NA.IG sampled on December 17, 2007 and the iTraxx Europe series sampled on March 31, 2008. The CDS spreads are sorted in decreasing order.

Table 1: CDX.NA.IG Series 9, December 17, 2007 and iTraxx Europe Series 9, March 31, 2008. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The $[0, 3]$ spread is quoted in %. All maturities are for five years.

CDX 2007-12-17

Tranche	$[0,3]$	$[3,7]$	$[7,10]$	$[10,15]$	$[15,30]$
Market spread	48.07	254.0	124.0	61.00	41.00
Model spread	48.07	254.0	124.0	61.00	38.94
Absolute error in bp	0.010	0.000	0.000	0.000	2.061
Relative error in %	0.0001	0.000	0.000	0.000	5.027

iTraxx Europe 2008-03-31

Tranche	$[0,3]$	$[3,6]$	$[6,9]$	$[9,12]$	$[12,22]$
Market spread	40.15	479.5	309.5	215.1	109.4
Model spread	41.68	429.7	309.4	215.1	103.7
Absolute error in bp	153.1	49.81	0.0441	0.0331	5.711
Relative error in %	3.812	10.39	0.0142	0.0154	5.218

When calibrating the joint default intensities $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ for the CDX.NA.IG Series 9, December 17, 2007 we used 5 groups $I_1, I_2, \ldots, I_5$ where $I_j = \{1, \ldots, i_j\}$ for $i_j = 6, 19, 25, 61, 125$. Recall that we label the obligors by decreasing level of riskiness. We use the average over 3-year and 5-year CDS spreads as a measure of riskiness. Consequently, obligor 1 has the highest average CDS spread while company 125 has the lowest average CDS spread. Moreover, the obligors in the set $I_5 \setminus I_4$ consisting of the 64 safest companies are assumed to never default individually, and the corresponding CDSs are excluded from the calibration, which in turn relaxes the constraints for $\boldsymbol{\lambda}$ in (19). Hence, the obligors in $I_5 \setminus I_4$ can only bankrupt due to a simultaneous default of the companies in the group $I_5 = \{1, \ldots, 125\}$, i.e., in an Armageddon event. With this structure the calibration against the December 17, 2007 data-set is very good as can be seen in Table 1. By using stochastic recoveries specified as in [5] one can get a perfect fit of the same data-set, see in [5]. The calibrated common shock intensities $\boldsymbol{\lambda}$ for the 5 groups in the December 17, 2007 data-set are displayed in the left subplot of Figure 2.

The calibration of the joint default intensities $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ for the data sampled at March 31, 2008 is more demanding. This time we use 18 groups $I_1, I_2, \ldots, I_{18}$ where $I_j = \{1, \ldots, i_j\}$ for $i_j = 1, 2, \ldots, 11, 13, 14, 15, 19, 25, 79, 125$. In order to improve the fit, as in the 2007-case, we relax the constraints for $\boldsymbol{\lambda}$ in (19) by excluding from the calibration the CDSs corresponding to the obligors in $I_{18} \setminus I_{17}$. Hence, we assume that the obligors in $I_{18} \setminus I_{17}$ never default individually, but can only bankrupt due to an simultaneous default of all companies in the group $I_{18} = \{1, \ldots, 125\}$. In this setting, the calibration of the 2008 data-set with constant recoveries yields an acceptable fit except for the $[3,6]$ tranche, as can be seen in Table 1. However, by including stochastic recoveries specified as in [5] the fit can be substantially improved, see in [5]. Furthermore, the more groups added the better the fit, which explain why we use as many as 18 groups (this holds both for constant and stochastic recoveries). The calibrated common shock intensities $\boldsymbol{\lambda}$ for the 18 groups in the March 2008 data-set are displayed in the right subplot of Figure 2.

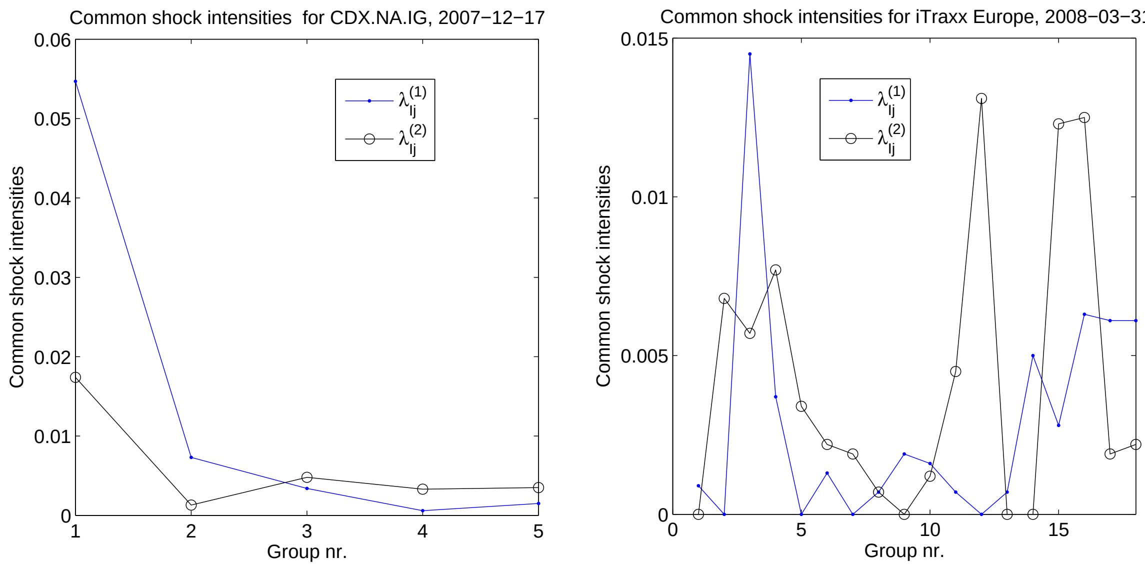


Figure 2: The calibrated common shock intensities $(\lambda^{(k)}_{I_j})_{j,k}$ for the two portfolios CDX.NA.IG sampled on December 17, 2007 (left) and the iTraxx Europe series sampled on March 31, 2008 (right).

Let us finally discuss the choice of the groupings $I_1 \subset I_2 \subset \ldots \subset I_m$ in our calibrations. First, for the CDX.NA.IG Series 9, December 17, 2007 data set, we used $m = 5$ groups with as always $i_m = n$. For $j = 1, 2$ and 4 the choice of i_j corresponds to the number of defaults needed for the loss process with constant recovery of 40% to reach the j-th attachment points. Hence, $i_j \cdot \frac{1-R}{n}$ with $R = 40\%$ and $n = 125$ then approximates the attachment points $3\%, 10\%, 30\%$ which explains the choice $i_1 = 6, i_2 = 19, i_4 = 61$. The choice of $i_3 = 25$ implies a loss of 12% and gave a better fit than choosing i_3 to exactly match 15%. Finally, no group was chosen to match the attachment point of 7% since this made the calibration worse off for all groupings we tried. With the above grouping structure we got almost perfect fits in the constant recovery case as was seen in Table 1 (and a perfect fit with stochastic recovery, see in [5]). Unfortunately, using the same technique on the market CDO data from the iTraxx Europe series sampled on March 31, 2008 was not enough to achieve good calibrations. Instead more groups had to be added and we tried different groupings which led to the optimal choice rendering the calibration in Table 1. To this end, it is of interest to study the sensitivity of the calibrations with respect to the choice of the groupings on the form $I_1 \subset I_2 \subset \ldots \subset I_m$ where $I_j = \{1, \ldots, i_j\}$ for $i_j \in \{1, 2, \ldots, m\}$ and $i_1 < \ldots < i_m = 125$ on the March 31, 2008, data set. Three such groupings are displayed in Table 2 and the corresponding calibration results on the 2008 data set is showed in Table 3.

Table 2: Three different groupings (denoted A,B and C) consisting of $m = 7, 9, 13$ groups having the structure $I_1 \subset I_2 \subset \ldots \subset I_m$ where $I_j = \{1, \ldots, i_j\}$ for $i_j \in \{1, 2, \ldots, m\}$ and $i_1 < \ldots < i_m = 125$.

Three different groupings

i_j	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9	i_{10}	i_{11}	i_{12}	i_{13}
Grouping A	6	14	15	19	25	79	125						
Grouping B	2	4	6	14	15	19	25	79	125				
Grouping C	2	4	6	8	9	10	11	14	15	19	25	79	125

Table 3: The relative calibration error in percent of the market spread, for the three different groupings A, B and C in Table 2, when calibrated against CDO tranche on iTraxx Europe Series 9, March 31, 2008 (see also in Table 1).

Relative calibration error in % (constant recovery)

Tranche	$[0, 3]$	$[3, 6]$	$[6, 9]$	$[9, 12]$	$[12, 22]$
Error for grouping A	6.875	18.33	0.0606	0.0235	4.8411
Error for grouping B	6.622	16.05	0.0499	0.0206	5.5676
Error for grouping C	4.107	11.76	0.0458	0.0319	3.3076

From Table 3 we see that the relative calibration error in percent of the market spread decreased monotonically for the first three thranches as the number of groups increased. The rest of the parameters in the calibration where the same as in the optimal calibration in Table 1.

Finally, we remark that the two optimal groupings used in Table 1 in the two different data sets CDX.NA.IG Series 9, December 17, 2007 and iTraxx Europe Series 9, March 31, 2008 differ quite a lot. However, the CDX.NA.IG Series is composed by North American obligors while the iTraxx Europe Series is formed by European companies. Thus, there is no model risk or inconsistency created by using different groupings for these two different data sets, coming from two disjoint markets. If on the other hand the same series is calibrated and assessed (e.g. for hedging) at different time points in a short time span, it is of course desirable to use the same grouping in order to avoid model risk.

4.2.1 The Implied Loss Distribution

After the fit of the model against market spreads we can use the calibrated portfolio parameters $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ together with the calibrated individual default intensities, to study the credit-loss distribution in the portfolio. In this paper we only focus on some few examples derived from the loss distribution with constant recoveries evaluated at $T = 5$ years.

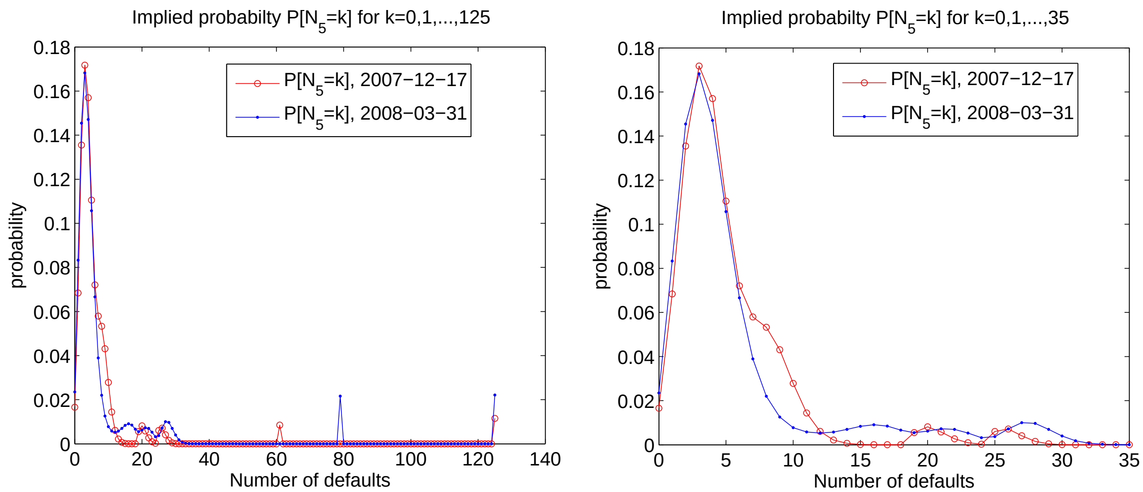


Figure 3: The implied distribution $\mathbb{P}[N_5 = k]$ on $\{0, 1, \ldots, \ell\}$ where $\ell = 125$ (left) and $\ell = 35$ (right) when the model is calibrated against CDX.NA.IG Series 9, December 17, 2007 and iTraxx Europe Series 9, March 31, 2008.

The allowance of joint defaults of the obligors in the groups I_j together with the restriction of the most safest obligors not being able to default individually, will lead to some interesting effects of the loss distribution, as can be seen in Figures 3 and 4. For example, we clearly see that the support of the loss-distributions will in practice be limited to a rather compact set. To be more specific, the graphs in Figure 3 indicate that $\mathbb{P}[N_5 = k]$ roughly has support on the set $\{1, \ldots, 35\} \cup \{61\} \cup \{125\}$ for the 2007 case and on $\{1, \ldots, 40\} \cup \{79\} \cup \{125\}$ for the 2008 data-set. This becomes even more clear in a log-loss distribution, as is seen in Figure 4.

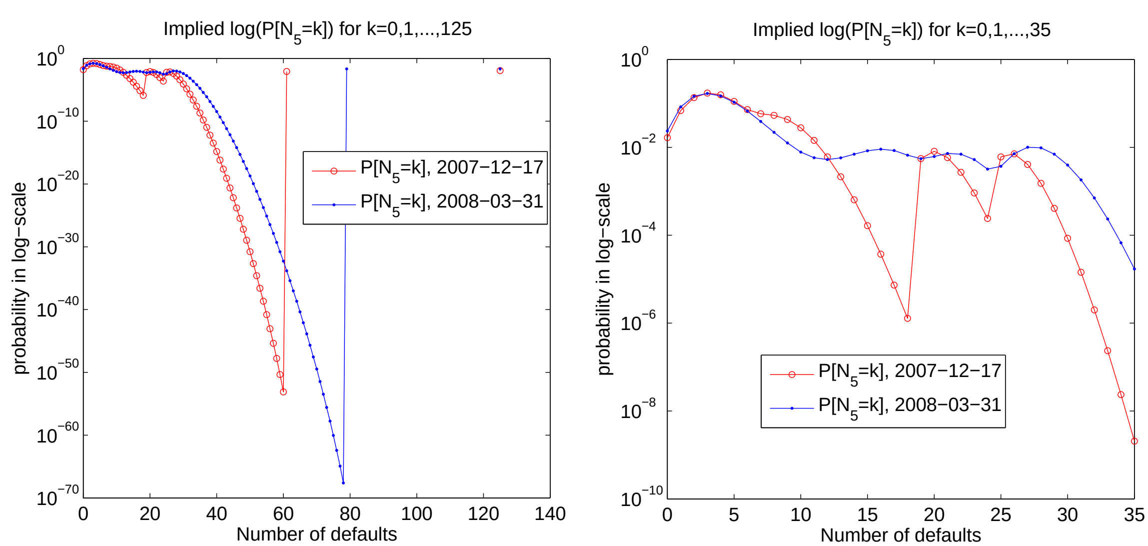


Figure 4: The implied log distribution $\ln(\mathbb{P}[N_5 = k])$ on $\{0, 1, \ldots, \ell\}$ where $\ell = 125$ (left) and $\ell = 35$ (right) when the model is calibrated against CDX.NA.IG Series 9, December 17, 2007 and iTraxx Europe Series 9, March 31, 2008.

From the left graph in Figure 4 we see that the default-distribution is nonzero on $\{36, \ldots, 61\}$ in the 2007-case and nonzero on $\{41, \ldots, 79\}$ for the 2008-sample, but the actual size of the loss-probabilities are in the range 10^{-10} to 10^{-70}. Such low values will obviously be treated as zero in any practically relevant computation. Furthermore, the reasons for the empty gap in the left graph in Figure 4 on the interval $\{62, \ldots, 124\}$ for the 2007-case is due to the fact that we forced the obligors in the set $I_5 \setminus I_4$ to never default individually, but only due to an simultaneous common shock default of the companies in the group $I_5 = \{1, \ldots, 125\}$. This Armageddon event is displayed as an isolated nonzero 'dot' at default nr 125 in the left graph of Figure 4. The gap on $\{80, \ldots, 124\}$ in the 2008 case is explained similarly due to our assumption on the companies in the set $I_{19} \setminus I_{18}$. Also note that the two 'dots' at default nr 125 in the left subplot of Figure 4 are manifested as spikes in the left graph displayed in Figure 3. The shape of the multimodal loss distributions presented in Figure 3 and Figure 4 are typical for models allowing simultaneous defaults, see for example Figure 2, page 59 in [12] and Figure 2, page 710 in [14].

5 Min-Variance Hedging

In this section we present some numerical results illustrating performance of the min-variance hedging strategies given in Proposition 3.2 of [4]. This will be done in the setup of the calibrated model of Subsection 4.1 (model calibrated with constant recoveries to the CDX.NA.IG Series 9 data set of December 17, 2007).

The aim of this subsection is to analyze the composition of the hedging portfolio at time $t = 0$ (the calibration date) when standardized CDO tranches are hedged with a group of d single-name CDSs, which are included in the underlying CDS index. Since no spread factor $\mathbf{X}$ is used in the model, Proposition 3.2 of [4] then implies that the min-variance hedging ratios at time $t = 0$ is given by $\zeta^{va}(0, \mathbf{H}_0) = (u, \mathbf{v})(\mathbf{v}, \mathbf{v})^{-1}(0, \mathbf{H}_0)$ where

$$(u, \mathbf{v}) = \sum_{Y \in \mathcal{Y}} \lambda_Y(0) \Delta u^Y (\Delta \mathbf{v}^Y)^\mathsf{T} \quad \text{and} \quad (\mathbf{v}, \mathbf{v}) = \sum_{Y \in \mathcal{Y}} \lambda_Y(0) \Delta \mathbf{v}^Y (\Delta \mathbf{v}^Y)^\mathsf{T}.$$

Hence, computing the min-variance hedging ratios involves a summation of the "jump differentials" $\lambda_Y(0)\Delta u^Y (\Delta \mathbf{v}^Y)^\mathsf{T}$ and $\lambda_Y(0)\Delta \mathbf{v}^Y (\Delta \mathbf{v}^Y)^\mathsf{T}$ over all possible triggering events $Y \in \mathcal{Y}$ where $\mathcal{Y} = \{\{1\}, \ldots, \{n\}, I_1, \ldots, I_m\}$.

In the calibration of the CDX.NA.IG Series 9, we used $m = 5$ groups $I_1, I_2, \ldots, I_5$ where $I_j = \{1, \ldots, i_j\}$ for $i_j = 6, 19, 25, 61, 125$ and the obligors have been labeled by decreasing level of riskiness. At the calibration date $t = 0$ associated with December 17, 2007, no name has defaulted in CDX Series 9 so we set $\mathbf{H}_0 = \mathbf{0}$. In our empirical framework, the intensities $\lambda_Y(0)$, $Y \in \mathcal{Y}$ are computed from the constant default intensities λ_i that fit market spreads of 3-year maturity CDSs and from the 3-year horizon joint default intensities λ_{I_j} calibrated to CDO tranche quotes. The terms $\Delta u^Y(0, \mathbf{H}_0)$ and $\Delta \mathbf{v}^Y(0, \mathbf{H}_0)$ corresponds to the change in value of the tranche and the single-name CDSs, at the arrival of the triggering event affecting all names in group Y. Recall that the cumulative change in value of the tranche is equal to

$$\Delta u^Y(0, \mathbf{H}_0) = L_{a,b}(\mathbf{H}_0^Y) - L_{a,b}(\mathbf{H}_0) + u(0, \mathbf{H}_0^Y) - u(0, \mathbf{H}_0)$$

where $\mathbf{H}_0^Y$ is the vector of $\{0, 1\}^n$ such that only the components $i \in Y$ are equal to one. Hence, the tranche sensitivity $\Delta u^Y(0, \mathbf{H}_0)$ includes both the protection payment on the tranche associated with the default of group Y and the change in the ex-dividend price u of the tranche. Note that the price sensitivity is obtained by computing the change in the present value of the default leg and the premium leg. The latter quantity involves the contractual spread that defines cash-flows on the premium leg. As for CDX.NA.IG Series 9, the contractual spreads were equal to 500 bps, 130 bps, 45 bps, 25 bps and 15 bps for the tranches [0-3%], [3-7%], [7-10%], [10-15%] and [15-30%]. We use the common-shock interpretation to compute $u(0, \mathbf{H}_0^Y)$ and $u(0, \mathbf{H}_0)$ with the convolution recursion pricing scheme detailed in Subsection 3. More precisely, using the same notation as in Subsection 3, the CDO tranche price $u(0, \mathbf{H}_0^Y)$ (resp. $u(0, \mathbf{H}_0)$) is computed using the recursion procedure with $Z = N_n \setminus Y$ (resp. $Z = N_n$). We let $i_1, \ldots i_d$ be the CDSs used in the min-variance hedging and assume that they all are initiated at time $t = 0$. Hence, the market value at $t = 0$ for these CDSs are zero. As a result, when group Y defaults simultaneously, the change in value $\Delta\mathbf{v}^Y(0, \mathbf{H}_0)$ for buy-protection positions on these CDSs is only due to protection payment associated with names in group Y. Hence, for one unit of nominal exposure on hedging CDSs, the corresponding vector of sensitivities is equal to $\Delta\mathbf{v}^Y(0, \mathbf{H}_0) = ((1-R)\mathbb{1}_{i_1 \in Y}, \ldots, (1-R)\mathbb{1}_{i_d \in Y})^\mathsf{T}$ where the recovery rate R is assumed to be constant and equal to 40%.

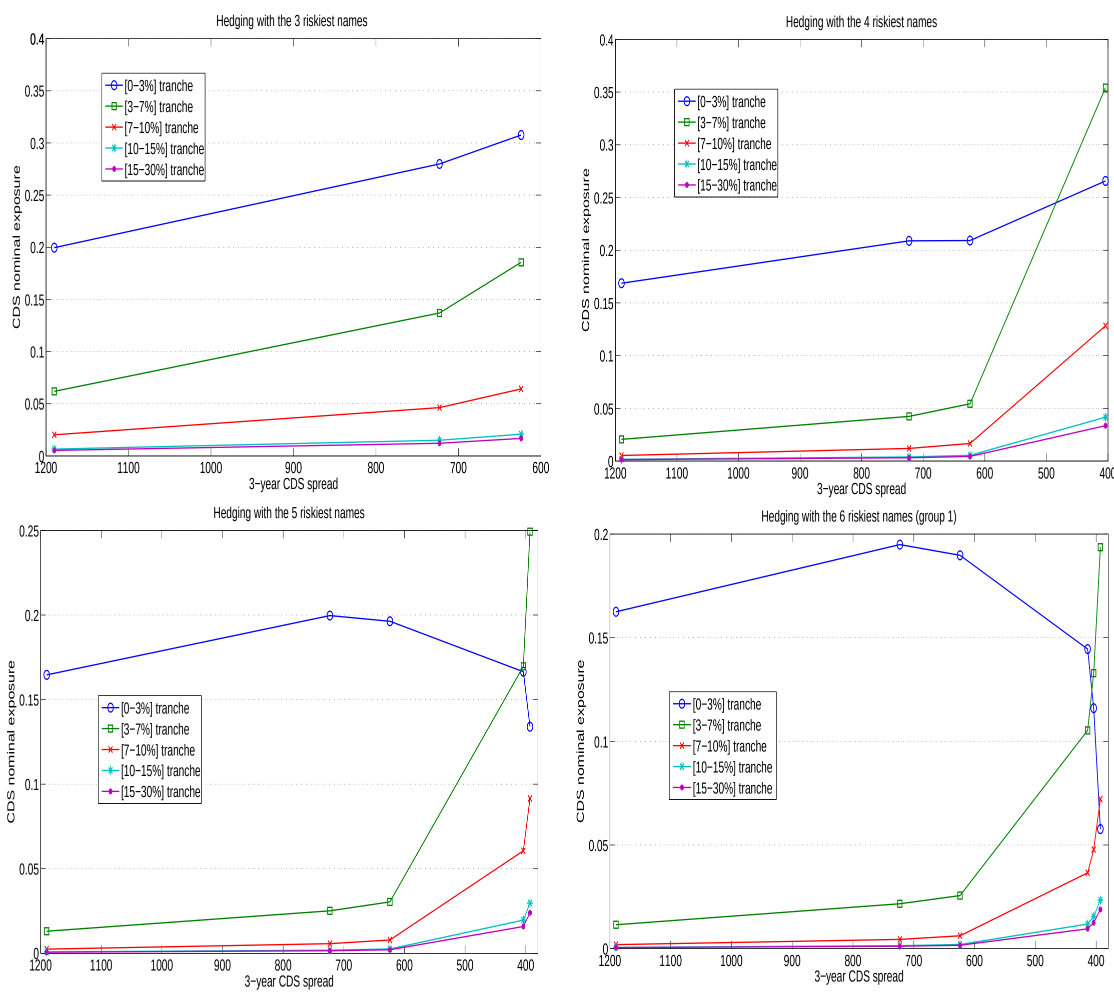


Figure 5: Min-variance hedging strategies associated with the d riskiest CDSs, $d = 3, 4, 5, 6$ for one unit of nominal exposure of different CDO tranches in a model calibrated to market spreads of CDX.NA.IG Series 9 on December 17, 2007.

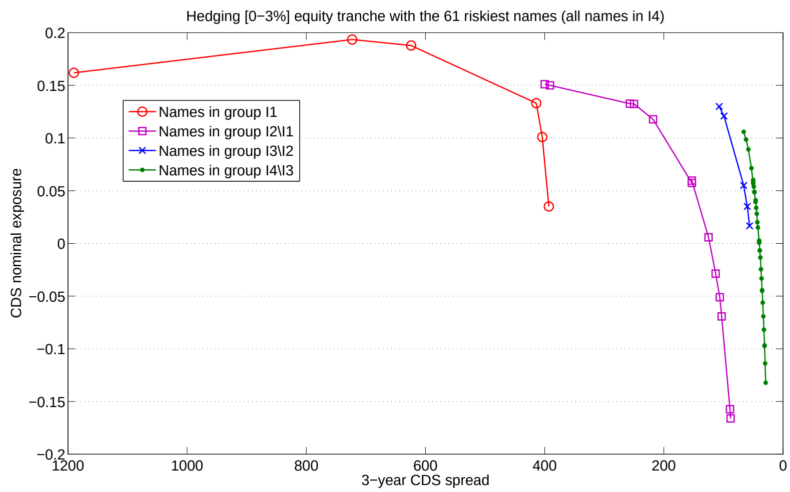
Table 4: The names and CDS spreads (in bp) of the six riskiest obligors used in the hedging strategy displayed by Figure 5.

Company (Ticker)	CCR-HomeLoans	RDN	LEN	SFI	PHM	CTX
3-year CDS spread	1190	723	624	414	404	393

Figure 5 displays the nominal exposure for the d most riskiest CDSs when hedging one unit of nominal exposure in a CDO by using the min-variance hedging strategy in Proposition 3.2 of [4]. We use $d = 3, 4, 5$ and $d = 6$ in our computations. Furthermore, Table 4 displays the names and sizes of the 3-year CDS spreads used in the hedging strategy. Each plot in Figure 5 should be interpreted as follows: in every pair (x, y) the x-component represents the size of the 3-year CDS spread at the hedging time $t = 0$ while the y-component is the corresponding nominal CDS-exposure computed via Proposition 3.2 of [4] using the d riskiest CDSs. The graphs are ordered from top to bottom, where the top panel corresponds to hedging with the $d = 3$ riskiest CDS and the bottom panel corresponds to hedging with the $d = 6$ riskiest names. Note that the x-axes are displayed from the riskiest obligor to the safest. Thus, hedge-sizes y for riskier CDSs are aligned to the left in each plot while y-values for safer CDSs are consequently displayed more to the right. In doing this, going from the top to the bottom panel consists in observing the effect of including new safer names from the right part of the graphs. We have connected the pairs (x, y) with lines forming graphs that visualizes possible trends of the min-variance hedging strategies for the d most riskiest CDSs.

For example, when the three riskiest names are used for hedging (top panel), we observe that the amount of nominal exposure in hedging instruments decreases with the degree of subordination, i.e., the [0-3%] equity tranche requires more nominal exposure in CDSs than the upper tranches. Note moreover that the min-variance hedging portfolio contains more CDSs on names with lower spreads. When lower-spread CDSs are added in the portfolio, the picture remains almost the same for the 3 riskiest names. For the remaining safer names however, the picture depends on the characteristics of the tranche. For the [0-3%] equity tranche, the quantity of the remaining CDSs required for hedging sharply decrease as additional safer names are added. One possible explanation is that adding too many names in the hedging strategy will be useless when hedging the equity tranche. This is intuitively clear since one expects that the most riskiest obligors will default first and consequently reduce the equity tranche substantially, explaining the higher hedge-ratios for riskier names, while it is less likely that the more safer names will default first and thus incur losses on the first tranche which explains the lower hedge ratios for the safer names. We observe the opposite trend for the senior (safer) tranches: adding new (safer) names in the hedging portfolio seems to be useful for "non equity" tranches since the nominal exposure required for these names increases when they are successively added.

Figure 6 and 7 display min-variance hedging strategies when hedging a standard tranche with the 61 riskiest names, i.e., all names excepted names in group $I_5 \setminus I_4$. Contrary to Figure 5, these graphs allow to visualize the effect of the "grouping structure" on the composition of the hedging portfolio. In this respect, we use different marker styles in order to distinguish names in the different disjoint groups I_1, $I_2 \setminus I_1$, $I_3 \setminus I_2$, $I_4 \setminus I_3$. As one can see, the min-variance hedging strategies are quite different among tranches. Moreover, whereas nominal exposures required for hedging are clearly monotone for names belonging to the same disjoint group, this tendency is broken when we consider names in different groups. This suggests that the grouping structure has a substantial impact on the distribution of names in the hedging portfolio. For the equity tranche, we observe in Figure 5 that less safer-names are required for hedging. This feature is retained in Figure 6 when we look at names in specific disjoint groups. Indeed, names in a given disjoint group are affected by the same common-shocks which in turn affect the equity tranche with the same severity. The only effect that may explain differences in nominal exposure among names in the same disjoint group is spontaneous defaults: names with wider spreads are more likely to default first, then we need them in greater quantity for hedging than names with tighter spreads.



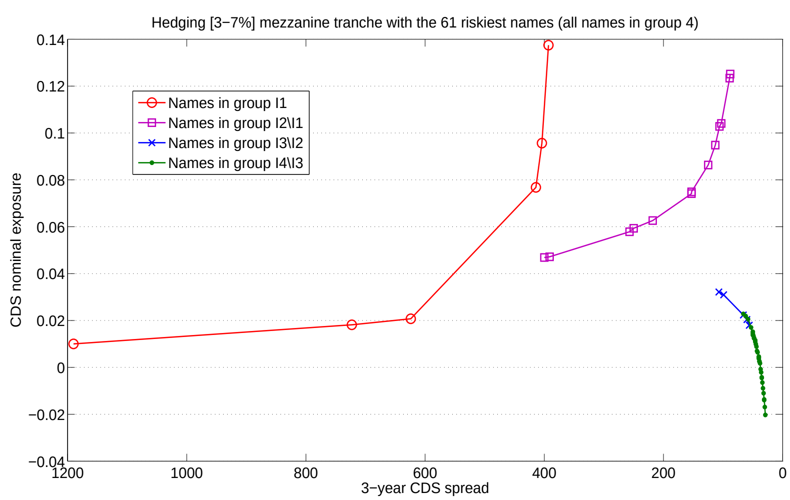


Figure 6: Min-variance hedging strategies when hedging one unit of nominal exposure in the [0-3%] equity tranche (top) and the [3-7%] mezzanine tranche (bottom) using the d riskiest CDSs, $d = 61$ (all names excepted names in group $I_5 \setminus I_4$) for one unit of nominal exposure.

Note that nominal exposure in hedging CDS even becomes negative for names within groups $I_2 \setminus I_1$ and $I_4 \setminus I_3$ when spreads are low. However, in Figure 6 we observe that, for the equity tranche, some of the riskiest names in $I_4 \setminus I_3$ are more useful in the hedging

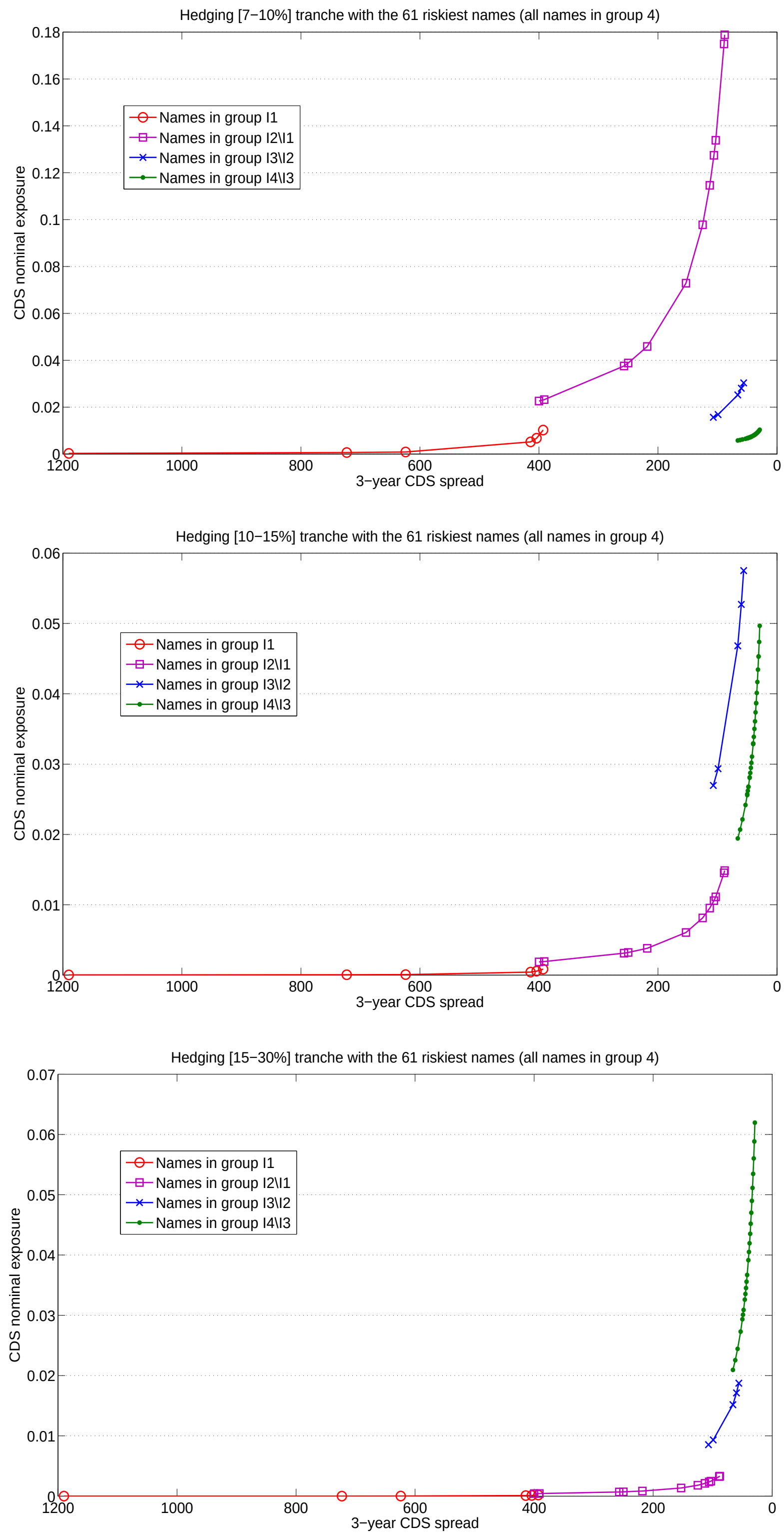


Figure 7: Min-variance hedging strategies when hedging one unit of nominal exposure in the [7-10%] tranche (top), the [10-15%] tranche (middle) and the [15-30%] tranche (bottom) with the d riskiest CDSs, $d = 61$ (all names excepted names in group $I_5 \setminus I_4$).

than some of the safest names in group I_1, which may sound strange at a first glance, given that the credit spread of the latter is much larger than the credit spread of the former. Recall that the equity tranche triggers protection payments corresponding to the few first defaults, if these occur before maturity. Even if names in group $I_4 \backslash I_3$ have a very low default probability, the fact that they can affect the tranche at the arrival of common-shocks I_4 or I_5 makes these names appealing for hedging because they are less costly (they require less premium payments) than names in I_1.

Figure 6 suggests that names with the lowest spreads should be ineffective to hedge the [0-3%] and the [3-7%] tranches. As can be seen in Figure 7, this is the contrary for the other tranches, i.e., the amount of low-spread names in the hedging portfolio increases as the tranche becomes less and less risky. For the [15-30%] super-senior tranche, we can see on the lowest graph of Figure 7 that the safer a name is, the larger the quantity which is required for hedging. Furthermore, Figure 7 also shows that in a consistent dynamic model of portfolio credit risk calibrated to a real data set, the [15-30%] super-senior tranche has significant (in fact, most of its) sensitivity to very safe names with spreads less than a few dozens of bp-s. For this tranche it is actually likely that one could improve the hedge by inclusion of even safer names to the set of hedging instruments, provided these additional names could also be calibrated to. Recall that on the data of CDX.NA.IG Series 9 on December 17, 2007, we calibrated our model to the 64 safest names in the portfolio.

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