Hedging CDO tranches in a Markovian environment

Areski Cousin, Monique Jeanblanc, Jean-Paul Laurent

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Contents

Hedging CDO tranches in a Markovian environment 1
Areski Cousin, Monique Jeanblanc, Jean-Paul Laurent

Part I Hedging CDO tranches in a Markovian environment

1 Hedging instruments 5
Areski Cousin, Monique Jeanblanc, Jean-Paul Laurent
1.1 Credit Default Swap 5
1.1.1 Cash-flow description 6
1.2 Theoretical Framework 7
1.2.1 Default times 7
1.2.2 Market assumptions 8
1.2.3 Hedging and martingale representation theorem 9
1.3 The single default case 13
1.3.1 Some important martingales 13
1.3.2 CDS market value 15
1.3.3 CDS market Spreads 16
1.3.4 Dynamics of CDS Prices in a single default setting 17
1.4 Two default times 18
1.4.1 Intensities 19
1.4.2 Dynamics of CDS prices in a two defaults setting 21

2 Hedging Default Risks of CDOs in Markovian Contagion Models . . . 25
Areski Cousin, Monique Jeanblanc, Jean-Paul Laurent
2.1 Synthetic CDO tranches 27
2.1.1 Credit Default Swap Indices 28
2.1.2 Standardized CDO tranches 28
2.1.3 Cash-flows description 29
2.1.4 CDO tranche price and market spread 32
2.2 Homogeneous Markovian contagion models 33
2.2.1 Intensity specification 34

2.2.2 Risk-neutral pricing . . . 35
2.2.3 Computation of credit deltas . . . 36
2.3 Calibration of loss intensities . . . 38
2.3.1 Calibration of loss intensities on a complete set of number of defaults probabilities . . . 39
2.3.2 Calibration of time homogeneous loss intensities . . . 40
2.3.3 Other calibrating approaches . . . 41
2.4 Computation of credit deltas through a recombining tree . . . 42
2.4.1 Building up a tree . . . 42
2.4.2 Computation of hedge ratios for CDO tranches . . . 44
2.4.3 Model calibrated on a loss distribution associated with a Gaussian copula . . . 47
2.4.4 Dependence of hedging strategies upon the correlation parameter . . . 50
2.4.5 Model calibrated on a loss distribution associated with CDO tranche quotes . . . 52
2.4.6 Comparison with standard market practice . . . 56
2.4.7 Comparison with deltas computed in other dynamic credit risk models . . . 58

3 Conclusion . . . 61
Areski Cousin, Monique Jeanblanc, Jean-Paul Laurent

References . . . 65

Part I
Hedging CDO tranches in a Markovian environment

The risk management and the hedging of credit derivatives and related products are topics of tremendous importance, especially given the recent credit turmoil. The risks at hand are usually split into different categories, which may sometimes overlap, such as credit spread and default risks, correlation and contagion risks. The credit crisis also drove attention to counterparty risk and related issues such as collateral management, downgrading of guarantors and of course liquidity issues. For simplicity, these will not be dealt within this part[1].

Credit derivatives are over-the-counter (OTC) financial instruments designed to transfer credit risk of a reference entity between two counterparties by way of a bilateral agreement. The agreement involves a seller of protection and a buyer of protection. The seller of protection is committed to cover the losses induced by the default of a reference entity, typically a corporate. In return, the buyer of protection has to pay at some fixed dates a premium to the seller of protection. By the default, we mean that the entity goes bankrupt or fails to pay a coupon on time, for some of its issued bonds. Even though credit derivatives are traded over-the-counter, credit events are standardized by the International Swap and Derivative Association (ISDA)[2].

Since credit derivatives involve some counterparty risk, the protection seller may be asked to post some collateral. Also, depending on the market value of the contract, the amount of collateral may be dynamically adjusted. However, after the recent credit crisis and subsequent defaults, settlement procedures had to be updated. Various projects including the ISDA, tend to standardize the cash-flows of CDS, netting and settlement procedures. It is likely that some market features will change. Nevertheless, the main ideas expressed here will still be valid with some minor adaptation.

Financial institutions such as banks, mutual funds, pension funds, insurance and reinsurance companies, monoline insurance companies, corporations or sovereign wealth funds have a natural incentive to use credit derivatives in order to assume, reduce or manage credit exposures.

Surprisingly enough, since pricing at the cost of the hedge is the cornerstone of the derivatives modelling field, models that actually connect pricing and hedging issues for CDOs have been studied after the one factor Gaussian copula model became a pricing standard. This discrepancy with the equity or interest derivatives fields can actually be seen as a weakness and one can reasonably think that further

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[1] See [33] for a discussion of the issues involved.

[2] Although ISDA reports a list of six admissible credit events, most of the contracts only include bankruptcy and failure to pay as credit events. This is the case of contracts referencing companies settled in developed countries. The definitions have been last updated in 2003. An overview of these standardized definitions can be found in [54]. However, these are likely to be updated, for instance due to the ISDA big bang protocol.

researches in the credit area will aim at closing the gap between pricing and hedging.

Before proceeding further, let us recall the main features in a hedging and risk management problem, which come to light whatever the underlying risks:

- A first issue is related to the choice and the liquidity of the hedging instruments: typically, one could think of credit index default swaps, CDS on names with possibly different maturities, standardized synthetic single tranche CDOs and even other products such as equity put options, though this will not be detailed in this partq. We reckon that the use of equity products to mitigate risks can be useful in the high yield market, but this is seemingly not the case for CDO tranches related to investment grade portfolios.

- A second issue is related to the products to be hedged. In the remainder, we will focus either on single name CDS or basket credit derivatives, such as First to Default Swaps, CDO tranches, bespoke CDOs or tranchelets. We will leave aside interest rate or foreign exchange hybrid products, credit spread options and exotic basket derivatives such as leveraged tranches, forward starting CDOs or tranche options.

- A third issue relies on the choice of the hedging method. The mainstream theoretical approach in mathematical finance favours the notion of replication of complex products through dynamic hedging strategies based on plain underlying instruments. However, it is clear that in many cases, risk can be mitigated by offsetting long and short positions, providing either a complete clearing or more usually leaving the dealer exposed to some basis albeit small risk. Moreover, such an approach is obviously quite robust to model risk. Unfortunately, there are some imbalances in customer demand and investment banks can be left with rather large outstanding positions on parts of the capital structure that must be managed up to maturity.

Chapter 1
Hedging instruments

Areski Cousin, Monique Jeanblanc, Jean-Paul Laurent

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This chapter is a primer about hedging of defaultable securities. It aims at presenting a general model of prices and hedging of defaultable claims, in a pure jump setting (there is no Brownian motion involved in our presentation). It also introduces the main hedging instruments we will consider throughout this part. We will particularly describe the cash-flows of credit default swaps (CDS) and derive the dynamics of their price. We also stress the impact of a credit event on the price dynamics of the surviving names.

1.1 Credit Default Swap

A Credit Default Swaps (CDS) is a bilateral over-the-counter agreement which transfers the credit risk of a defined reference entity from a buyer of protection to a seller of protection up to a fixed maturity time T. The reference entity denoted **C** is typically a corporate or a sovereign obligor.

We assume that **C** may default at a particular time τ which is a non negative random variable constructed on a probability space $(\Omega, \mathscr{G}, \mathbb{P})$. The default time τ corresponds to a credit event leading to payment to the protection buyer. Moreover, if **C** defaults, only a fraction R (the recovery rate) of the initial investment is recovered. Figure 1.1 illustrates the structure of a CDS.

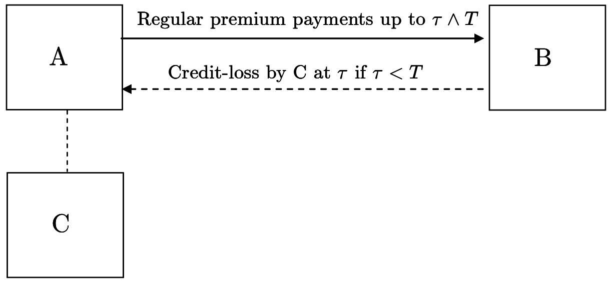

Fig. 1.1 Structure of a credit default swap

1.1.1 Cash-flow description

Let us consider a CDS initiated at time $t = 0$ with maturity T and nominal value E. The cash-flows of a CDS can be divided in two parts (or legs): the default leg which corresponds to the cash-flows generated by the seller of protection and the premium leg which is the set of cash-flows generated by the protection buyer. For simplicity, we will assume that nor the protection seller, neither the protection buyer can default.

Default leg

The seller of credit protection (denoted **B** in Figure 1.1) agrees to cover losses induced by the default of the obligor **C** at time τ if the latter occurs before maturity ($\tau < T$). In that case, the payment is exactly equal to the fraction of the loss that is not recovered, i.e., the loss given default $E(1-R)$. The settlement procedures in order to determine the recovery rate are not detailed here. The contract is worthless after the default of **C**.

Premium leg or fee leg

In return, the buyer of protection (denoted **A** in Figure 1.1) pays a periodic fee to **B** up to default time τ or until maturity T, whichever comes first. Each premium payment is proportional to a contractual credit spread[1] κ and to the nominal value E. More precisely, the protection buyer pays $\kappa \cdot \Delta_i \cdot E$ to the protection seller **B**, at every premium payment date $0 < T_1 < \cdots < T_p = T$ or until $\tau < T$, where $\Delta_i = T_i - T_{i-1}$, $i = 1, \ldots, p$ are the time intervals between two premium payment dates[2]. Let us remark that premium payments are made in arrears and begin at the end of the fist period (at T_1). If default happens between two premium payment dates, say $\tau \in]T_{i-1}, T_i[$, the protection fee has not been paid yet for the period $]T_{i-1}, \tau]$. In that case **A** will pay **B** an accrued premium equal to $\kappa \cdot (\tau - T_{i-1}) \cdot E$. The accrued pre-

[1] The contractual spread is quoted in basis points per annum.

[2] with the convention that $T_0 = 0$.

mium payment is usually made at time τ. After default of C ($t > \tau$), there are no more cash-flows on the premium leg which is worthless.

It is noteworthy that the contractual spread κ is fixed at inception (at $t = 0$) and remains the same until maturity. It is determined so that the expected discounted cash-flows (under a proper pricing measure to be detailed below) between **A** and **B** are the same when the CDS contract is settled.

Due to the credit turmoil, some major market participants encourage a change in market convention for single name CDS quotes. In the proposal, the contractual spread will be fixed at $\kappa = 100$ bps or $\kappa = 500$ bps depending on the quality of the credit. The buyer of protection will have to make an immediate premium payment (upfront payment) to enter the contract (see [5] for more details).

1.2 Theoretical Framework

1.2.1 Default times

In what follows, we consider n default times τ_i, $i = 1, \ldots, n$, that is, non-negative and finite random variables constructed on the same probability space $(\Omega, \mathscr{G}, \mathbb{P})$. For any $i = 1, \ldots, n$, we denote by $(N_t^i = \mathbb{1}_{\tau_i \leq t}, t \geq 0)$ the i-th default process, and by $\mathscr{H}_t^i = \sigma(N_s^i, s \leq t)$ the natural filtration of N^i (after completion and regularization on right). We introduce $\mathbb{H}$, the filtration generated by the processes N^i, $i = 1, \ldots, n$, defined as $\mathbb{H} = \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^n$, i.e., $\mathscr{H}_t = \vee_{i=1}^n \mathscr{H}_t^i$.
We denote by $\tau_{(1)}, \ldots, \tau_{(n)}$ the ordered default times.

Hypothesis 1: We assume that no simultaneous defaults can occur, i.e., $\mathbb{P}(\tau_i = \tau_j) = 0, \forall i \neq j$. This assumption is important with respect to the completeness of the market. As shown below, it allows to dynamically hedge credit derivatives referencing a pool of defaultable entities with n credit default swaps[3].

Hypothesis 2: We assume that, for any $i = 1, \ldots, n$, there exists a non-negative $\mathbb{H}$-adapted process $(\alpha_t^{i,\mathbb{P}}, t \geq 0)$ such that the process

$$M_t^{i,\mathbb{P}} := N_t^i - \int_0^t \alpha_s^{i,\mathbb{P}} ds \tag{1.1}$$

is a $(\mathbb{P}, \mathbb{H})$-martingale. The process $\alpha^{i,\mathbb{P}}$ is called the $(\mathbb{P}, \mathbb{H})$-intensity of τ_i (Note that the value of the intensity depends strongly of the underlying probability). This process vanishes after τ_i (otherwise, after τ_i, the martingale $M^{i,\mathbb{P}}$ would be continuous

[3] In the general case where multiple defaults could occur, we have to consider possibly 2^n states, and we would require non standard credit default swaps with default payments conditionally on all sets of multiple defaults to hedge multiname credit derivatives.

and strictly decreasing, which is impossible) and can be written $\alpha_t^{i,\mathbb{P}} = (1-N_t^i)\widetilde{\alpha}_t^{i,\mathbb{P}}$ for some $\mathbb{H}^1 \vee \cdots \vee \mathbb{H}^{i-1} \vee \mathbb{H}^{i+1} \vee \cdots \vee \mathbb{H}^n$-adapted process $\widetilde{\alpha}^{i,\mathbb{P}}$ (see [8] for more details). In particular, for $n = 1$, the process $\widetilde{\alpha}^{1,\mathbb{P}}$ is deterministic. In terms of the process $\widetilde{\alpha}^{i,\mathbb{P}}$, one has

$$M_t^{i,\mathbb{P}} = N_t^i - \int_0^{t\wedge\tau_i} \alpha_s^{i,\mathbb{P}} ds = N_t^i - \int_0^t (1-N_s^i)\widetilde{\alpha}_s^{i,\mathbb{P}} ds\,.$$

Comments:(a) Let us remark that the latter hypothesis is not as strong as it seems to be. Indeed, the process N^i is an increasing $\mathbb{H}$-adapted process, hence an $\mathbb{H}$-submartingale. The Doob-Meyer decomposition implies that there exists a unique increasing $\mathbb{H}$-predictable process Λ^i such that $(N_t^i - \Lambda_t^i, t \geq 0)$ is an $\mathbb{H}$-martingale. We do not enter into details here[4], it's enough to know that a left-continuous adapted process is predictable. It is also well known that the process Λ^i is continuous if and only if τ_i is totally inaccessible[5]. Here, we restrict our attention to processes Λ^i which are absolutely continuous with respect to Lebesgue measure.
(b) It will be important to keep in mind that the martingale $M^{i,\mathbb{P}}$ has only one jump of size 1 at time τ_i.

1.2.2 Market assumptions

For the sake of simplicity, let us assume that instantaneous digital default swaps are traded on the names. An instantaneous digital credit default swap on name i traded at time t is a stylized bilateral agreement between a buyer and a seller of protection. More precisely, the protection buyer receives one monetary unit at time $t+dt$ if name i defaults between t and $t+dt$. If α_t^i denotes the contractual spread of this stylized CDS, the seller of protection receives in return a fee equal to $\alpha_t^i dt$ which is paid at time $t+dt$ by the buyer of protection. The *cash-flows* associated with a buy protection position on an instantaneous digital default swaps on name i traded at time t are summarized in figure 1.2.

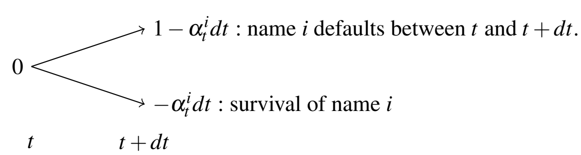

Fig. 1.2 *Cash-flows* of an instantaneous digital credit default swap (buy protection position)

[4] The reader is referred to [56] for the definition of a predictable process. A stopping time ϑ is predictable if there exists a sequence of stopping times ϑ_n such that $\vartheta_n < \vartheta$ and ϑ_n converges to ϑ as n goes to infinity.

[5] A stopping time τ is totally inaccessible if $\mathbb{P}(\tau = \vartheta) = 0$ for any predictable stopping time ϑ.

Let us also remark that there is no charge at inception (at time t) to enter an instantaneous digital credit default swap trade. Then, its payoff is equal to $dN_t^i - \alpha_t^i dt$ at $t + dt$ where dN_t^i is the payment on the default leg and $\alpha_t^i dt$ is the (short term) premium on the default swap.

Hypothesis 3: We assume that contractual spreads $\alpha^1, \ldots, \alpha^n$ are adapted to the filtration $\mathbb{H}$ of default times. The naturel filtration of default times can thus be seen as the relevant information on economic grounds.

Moreover, since the instantaneous digital credit default swap is worthless after default of name i, credit spreads must vanish after τ_i, i.e., $\alpha_t^i = 0$ on the set $\{t > \tau_i\}$.

Note that considering such instantaneous digital default swaps rather than actually traded credit default swaps is not a limitation of our purpose. This can rather be seen as a convenient choice of basis from a theoretical point of view.

For simplicity, we further assume that (continuously compounded) default-free interest rates are constant and equal to r. Given some initial investment V_0 and some $\mathbb{H}$-predictable bounded processes $\delta^1, \ldots, \delta^n$ associated with some self-financed trading strategy in instantaneous digital credit default swaps, we attain at time T the payoff:

$$V_0 e^{rT} + \sum_{i=1}^{n} \int_0^T \delta_s^i e^{r(T-s)} \left(dN_s^i - \alpha_s^i ds\right).$$

By definition, δ_s^i is the nominal amount of instantaneous digital credit default swap on name i held at time s. This induces a net cash-flow of $\delta_s^i \cdot \left(dN_s^i - \alpha_s^i ds\right)$ at time $s + ds$, which has to be invested in the default-free savings account up to time T.

1.2.3 Hedging and martingale representation theorem

In our framework (we do not have any extra noise in our model, and the intensities do no depend on an exogenous factor), individual default intensities are not driven by a specific spread risk but by the arrival of new defaults : default intensities $\alpha^{i,\mathbb{P}}$, $i = 1, \ldots, n$ are deterministic functions of the past default times between two default dates. More precisely, as we shall prove later on, the intensity of τ_i on the set $\{t; \tau_{(j)} \le t < \tau_{(j+1)}\}$ is a deterministic function of $\tau_{(1)}, \ldots, \tau_{(j)}$.

The main mathematical result of the study derives from the predictable representation theorem (see Theorem 9 in [11], Chapter III or [42]).

Theorem 1.1. *Let $A \in \mathscr{H}_T$ be a $\mathbb{P}$-integrable random variable. Then, there exists $\mathbb{H}$-predictable processes $\theta^i, i = 1, \ldots, n$ such that*

$$A = \mathbb{E}_{\mathbb{P}}[A] + \sum_{i=1}^{n} \int_0^T \theta_s^i \left(dN_s^i - \alpha_s^{i,\mathbb{P}} ds \right) = \mathbb{E}_{\mathbb{P}}[A] + \sum_{i=1}^{n} \int_0^T \theta_s^i dM_s^{i,\mathbb{P}} , \tag{1.2}$$

and $\mathbb{E}_{\mathbb{P}} \left(\int_0^T |\theta_s^i| \alpha_s^{i,\mathbb{P}} ds \right) < \infty$.

Proof. We do not enter into details. The idea is to prove that the set of random variables

$$Y = \exp \left(\sum_{i=1}^{n} \int_0^T \varphi_s^i dM_s^i - \int_0^T (e^{\varphi_s^i} - 1) \alpha_s^{i,\mathbb{P}} ds \right)$$

where φ^i are deterministic functions, is total in $L^2(\mathscr{H}_T)$ and to note that Y satisfies (1.2): indeed,

$$Y = 1 + \sum_{i=1}^{n} \int_0^T \varphi_s^i Y_{s-}^i dM_s^i .$$

Due to the integrability assumption on the r.v. A, and the predictable property of the θ's, the processes $\int_0^t \theta_s^i dM_s^i$, $i = 1, \ldots, n$ are $(\mathbb{P}, \mathbb{H})$-martingales. $\square$

Let us remark that relation (1.2) implies that the predictable representation theorem (PRT) holds: any $(\mathbb{P}, \mathbb{H})$-martingale can be written in terms of the fundamental martingales $M^{i,\mathbb{P}}$. Indeed, if $M^{\mathbb{P}}$ is a $(\mathbb{P}, \mathbb{H})$-martingale, applying (1.2) to $A = M_T^{\mathbb{P}}$ and using the fact that $\int_0^t \theta_s^i dM_s^{i,\mathbb{P}}$ are martingales,

$$M_t^{\mathbb{P}} = \mathbb{E}_{\mathbb{P}} \left[M_T^{\mathbb{P}} \mid \mathscr{H}_t \right] = \mathbb{E}_{\mathbb{P}} \left[M_T^{\mathbb{P}} \right] + \sum_{i=1}^{n} \int_0^t \theta_s^i dM_s^{i,\mathbb{P}}. \tag{1.3}$$

From the PRT, any strictly positive $(\mathbb{P}, \mathbb{H})$-martingale ζ with expectation equal to 1 (as any Radon-Nikodym density) can be written as

$$d\zeta_t = \zeta_{t-} \sum_{i=1}^{n} \theta_t^i dM_t^{i,\mathbb{P}}, \; \zeta_0 = 1. \tag{1.4}$$

Indeed, as any martingale, ζ admits a representation as

$$d\zeta_t = \sum_{i=1}^{n} \hat{\theta}_t^i dM_t^{i,\mathbb{P}}, \; \zeta_0 = 1$$

Since ζ is assumed to be strictly positive, introducing the predictable processes θ^i as $\theta_s^i = \frac{1}{\zeta_{s-}^i} \hat{\theta}_s^i$ allows to obtain the equality (1.4). We emphasize that the predictable property of θ is essential to guarantee that the processes $\int \theta_s dM_s^i$ are (local) martingales.

Conversely, the Doléans-Dade exponential, (unique) solution of

$$d\zeta_t = \zeta_{t-} \sum_{i=1}^{n} \theta_t^i dM_t^{i,\mathbb{P}}, \; \zeta_0 = 1$$

is a (local) martingale. Note that, in order that ζ is indeed a non-negative local martingale, one needs that $\theta_t^i > -1$. Indeed, the solution of (1.4) is

$$\zeta_t = \exp\left(-\int_0^t \sum_{i=1}^n \theta_s^i \alpha_s^{i,\mathbb{P}} ds\right) \prod_{i=1}^n (1+\theta_{\tau_i}^i)^{N_t^i} .$$

The process ζ is a true martingale under some integrability conditions on θ (e.g., θ bounded) or if $\mathbb{E}^{\mathbb{P}}[\zeta_t] = 1$ for any t. Note that the jump of ζ at time $t = \tau_i$ is $\Delta\zeta_t = \zeta_t - \zeta_{t-} = \zeta_{t-}\theta_t^i$ (so that $\zeta_t = \zeta_{t-}(1+\theta_t^i)$ at time τ_i, hence the condition on θ to preserve non-negativity of ζ).

Theorem 1.2. *Let ζ satisfying (1.4) with $\theta_t^i > -1$ and $\mathbb{E}^{\mathbb{P}}[\zeta_t] = 1$, and define the probability measure $\mathbb{Q}$ as*

$$d\mathbb{Q}|_{\mathscr{H}_t} = \zeta_t d\mathbb{P}|_{\mathscr{H}_t} .$$

Then, the process

$$M_t^i := M_t^{i,\mathbb{P}} - \int_0^t \theta_s^i \alpha_s^{i,\mathbb{P}} ds = N_t^i - \int_0^t (1+\theta_s^i)\alpha_s^{i,\mathbb{P}} ds$$

is a $\mathbb{Q}$-martingale. In particular, the $(\mathbb{Q},\mathbb{H})$-intensity of τ_i is $\alpha_t^i = (1+\theta_t^i)\alpha_t^{i,\mathbb{P}}$.

Proof. The process M^i is an $(\mathbb{Q},\mathbb{H})$-martingale if and only if the process $M^i\zeta$ is a $(\mathbb{P},\mathbb{H})$-martingale. Using integration by parts formula

$$\begin{aligned}
d(M_t^i\zeta_t) &= M_{t-}^i d\zeta_t + \zeta_{t-} dM_t^i + \Delta M_t^i \Delta\zeta_t \\
&= M_{t-}^i d\zeta_t + \zeta_{t-} dM_t^{i,\mathbb{P}} - \zeta_{t-}\theta_t^i \alpha_t^{i,\mathbb{P}} dt + \zeta_{t-}\theta_t^i dN_t^i \\
&= M_{t-}^i d\zeta_t + \zeta_{t-} dM_t^{i,\mathbb{P}} - \zeta_{t-}\theta_t^i \alpha_t^{i,\mathbb{P}} dt + \zeta_{t-}\theta_t^i (dM_t^{i,\mathbb{P}} + \alpha^{i,\mathbb{P}} dt) \\
&= M_{t-}^i d\zeta_t + \zeta_{t-} dM_t^{i,\mathbb{P}} - \zeta_{t-}\theta_t^i \alpha_t^{i,\mathbb{P}} dt + \zeta_{t-}\theta_t^i dM_t^{i,\mathbb{P}} + \zeta_{t-}\theta_t^i \alpha_t^{i,\mathbb{P}} dt \\
&= M_{t-}^i d\zeta_t + \zeta_{t-}\left(1+\theta_t^i\right) dM_t^{i,\mathbb{P}} .
\end{aligned}$$

□

Note that the predictable processes θ^i's used to define the Radon-Nikodym density in equation (1.4) can be chosen such that the instantaneous credit default swap spreads $\alpha^1, \ldots, \alpha^n$ (defined in Subsection 1.2.2) are exactly the $(\mathbb{Q},\mathbb{H})$-intensities associated with the default times. Let us recall that contractual spreads $\alpha^1, \ldots, \alpha^n$ are assumed to be adapted to the natural filtration $\mathbb{H}$ of default times. Moreover, from the absence of arbitrage opportunities, the cost of protection is positive if and only if a default risk exists. The latter argument implies that $\alpha^1, \ldots, \alpha^n$ are non negative $\mathbb{H}$-adapted processes and $\left\{\alpha_t^i > 0\right\} \overset{\mathbb{P}-a.s.}{=} \left\{\alpha_t^{i,\mathbb{P}} > 0\right\}$ for all time t and all name $i = 1, \ldots, n$. The processes θ^i, $i = 1, \ldots, n$ defined by

$$\theta_t^i = \left(\frac{\alpha_t^i}{\alpha_t^{i,\mathbb{P}}} - 1\right)(1 - N_{t-}^i),\ t \geq 0,\ i = 1, \ldots, n, \tag{1.5}$$

are therefore positive $\mathbb{H}$-predictable processes strictly greater than -1. They are admissible processes to define an equivalent change of probability measure. In the rest of the study, we will work under the probability $\mathbb{Q}$ obtained from $\mathbb{P}$ through the change of probability measure defined by equations (1.4) and (1.5).

It can be proved, using standard arguments that any $(\mathbb{Q},\mathbb{H})$-martingale can be written as a sum of integrals with respect to M^i. Let M be a $(\mathbb{Q},\mathbb{H})$-martingale. Then, there exists $\mathbb{H}$-predictable processes θ^i such that:

$$M_t = \mathbb{E}\left[M_T \mid \mathscr{H}_t\right] = \mathbb{E}\left[M_T\right] + \sum_{i=1}^{n} \int_0^t \theta_s^i dM_s^i, \tag{1.6}$$

where $\mathbb{E}$ is the expectation under $\mathbb{Q}$. Indeed, the process $(\widehat{M}_t := M_t \zeta_t, t \geq 0)$ being a $(\mathbb{P},\mathbb{H})$-martingale admits a representation as $\widehat{M}_t = M_0 + \sum_{i=1}^n \int_0^t \widehat{\theta}_s^i dM_s^{i,\mathbb{P}}$. It suffices to apply integration by parts formula to $M_t = \widehat{M}_t(\zeta_t)^{-1}$ to obtain the result.

In particular, for $A \in \mathscr{H}_T$, one has

$$A = \mathbb{E}\left[A \mid \mathscr{H}_t\right] + \sum_{i=1}^{n} \int_t^T \theta_s^i dM_s^i. \tag{1.7}$$

Starting from time t, we can thus replicate the claim A with the initial investment $V_t = \mathbb{E}\left[Ae^{-r(T-t)} \mid \mathscr{H}_t\right]$ (in the savings account) and the trading strategy based on instantaneous digital credit default swaps defined by $\delta_s^i = \theta_s^i e^{-r(T-s)}$ for $t \leq s \leq T$ and $i = 1, \ldots, n$. As there is no initial charge to enter an instantaneous digital credit default swap, $V_t = \mathbb{E}\left[Ae^{-r(T-t)} \mid \mathscr{H}_t\right]$ corresponds to the time-t replication price of A. Since A depends upon the default indicators of the names up to time T, this encompasses the cases of multiname credit derivatives such as CDO tranches and basket default swaps, provided that recovery rates are deterministic.

We can also remark that for a small time interval dt,

$$V_{t+dt} \approx V_t(1 + r dt) + \sum_{i=1}^{n} \delta_t^i \left(dN_t^i - \alpha_t^i dt\right) \tag{1.8}$$

which is consistent with market practice and regular rebalancing of the replicating portfolio. An investor who wants to be compensated at time t against the price fluctuations of A during a small period dt has to invest V_t in the risk-free asset and take positions $\delta^1, \ldots, \delta^n$ in the n instantaneous digital credit default swaps.

Thanks to the predictable representation theorem, it is also possible to describe the dynamics of a traditional credit default swap in terms of the dynamics of instantaneous credit default swaps. In the rest of this chapter we propose to build a general model of default times from the risk-neutral probability $\mathbb{Q}$ under which any defaultable claim can be replicated using instantaneous credit default swaps. There are various ways to construct such models. One of them, may be the most general in the case of non common defaults, is to start with the joint law of default times, and to make some regularity assumptions on that law (more precisely, that $G(t_1,\dots,t_n) := \mathbb{Q}(\tau_1 > t_1,\dots,\tau_i > t_i,\dots,\tau_n > t_n)$ is n-time differentiable with respect to $(t_1,\dots,t_n)$ and such that G and its derivatives do not vanish). We shall present this approach below, which is closely related to the well-known copula approach. Another way, more tractable but less general, is to specify the form of the intensities (in a Markov setting), and to construct the default times from these intensities. This approach will be presented in the following chapter and may be connected in some cases to the Markov chain used as a first step in a class of top-down models. A third method is to construct the random times as the first passage times at a random level for an increasing process. This last method is interesting for simulation, and allows correlation between the default times, via correlation of the random levels (see [59]). These three approaches both allow to derive the individual CDS spread dynamics as well as the dynamics of the portfolio loss, which will be needed for the pricing and hedging of CDO tranches.

In a first part, we shall present computations in the case $n = 1$. Then, we shall study the case $n = 2$.

1.3 The single default case

We study the case $n = 1$. Here, τ is a non-negative random variable on the probability space $(\Omega,\mathscr{G},\mathbb{Q})$ with risk-neutral survival function

$$G(t) := \mathbb{Q}(\tau > t) = 1-\mathbb{Q}(\tau \le t) = 1-F(t)$$

where F is the cumulative distribution function of τ, under $\mathbb{Q}$. We assume that $G(t) > 0, \forall t$, and that G is continuous. Here $\mathbb{H} = \mathbb{H}^1$.

1.3.1 Some important martingales

Lemma 1.3.1 *For any (integrable) random variable X*

$$\mathbb{E}(X|\mathscr{H}_t)\mathbb{1}_{t<\tau} = \mathbb{1}_{t<\tau}\frac{1}{G(t)}\mathbb{E}(X\mathbb{1}_{t<\tau}) \tag{1.9}$$

and for any Borelian (bounded) function h

$$\mathbb{E}(h(\tau)|\mathscr{H}_t) = \mathbb{1}_{\tau\le t}h(\tau) - \mathbb{1}_{t<\tau}\frac{1}{G(t)}\int_t^\infty h(u)dG(u)$$

Proof. This well known result is established in a more general setting in [21]. We give here a proof for completeness. For fixed t, the σ-algebra $\mathcal{H}_t$ being generated by the random variable $\tau \wedge t$, any $\mathcal{H}_t$-measurable random variable can be written as $h(\tau \wedge t)$ where h is a bounded Borel function. It is then obvious that, on the set $\{t < \tau\}$, any $\mathcal{H}_t$-measurable random variable is deterministic. Hence, there exists a constant k such that $\mathbb{E}(X|\mathcal{H}_t)\mathbb{1}_{t<\tau} = k\mathbb{1}_{t<\tau}$. Taking expectation of both members leads to $k = \frac{1}{G(t)}\mathbb{E}(X\mathbb{1}_{t<\tau})$. The second formula follows from

$$\mathbb{E}(h(\tau)|\mathcal{H}_t) = h(\tau)\mathbb{1}_{\tau \leq t} + \mathbb{1}_{t<\tau}\frac{\mathbb{E}(h(\tau)\mathbb{1}_{t<\tau})}{G(t)}$$

where we have used (1.9). The result is obtained with a computation of the last expectation. Note that the minus sign in front of the integral w.r.t. dG is due to the fact that G is decreasing. □

We now assume that G is differentiable (i.e., that τ admits a density f, so that $G'(t) = -f(t)$) (see [8] for the general case).

Proposition 1.3.1 *The process* $(M_t, t \geq 0)$ *defined as*

$$M_t = N_t - \int_0^{\tau\wedge t} \frac{f(s)}{G(s)}ds = N_t - \int_0^t (1-N_s)\frac{f(s)}{G(s)}ds$$

is an $\mathbb{H}$*-martingale. In other terms, the intensity of* τ *is* $(1-N_t)\widetilde{\alpha}(t)$ *where* $\widetilde{\alpha}$ *is the deterministic function* $\widetilde{\alpha}(t) = \frac{f(t)}{G(t)}$.

Proof. Let $s < t$. Then, from (1.9),

$$\mathbb{E}(N_t - N_s|\mathcal{H}_s) = \mathbb{1}_{\{s<\tau\}}\mathbb{E}(\mathbb{1}_{\{s<\tau\leq t\}}|\mathcal{H}_s) = \mathbb{1}_{\{s<\tau\}}\frac{F(t)-F(s)}{G(s)}\,. \tag{1.10}$$

On the other hand, the quantity

$$C := \mathbb{E}\left[\int_s^t (1-N_u)\frac{f(u)}{G(u)}du \,\Big|\, \mathcal{H}_s\right]$$

is equal to

$$\begin{aligned} C &= \int_s^t \frac{f(u)}{G(u)}\,\mathbb{E}\left[\mathbb{1}_{\{\tau>u\}}\big|\mathcal{H}_s\right] du = \mathbb{1}_{\{\tau>s\}}\int_s^t \frac{f(u)}{G(u)}\,\frac{G(u)}{G(s)}\,du \\ &= \mathbb{1}_{\{\tau>s\}}\frac{F(t)-F(s)}{G(s)} \end{aligned}$$

which, from (1.10), proves $\mathbb{E}(M_t - M_s|\mathcal{H}_s) = 0$, hence the desired result.

□

One should not confuse the intensity α and $\widetilde{\alpha}$, called the predefault-intensity. The intensity α is stochastic, and vanishes after τ, the predefault intensity is deterministic. The survival function G can be expressed in terms of the predefault intensity $\widetilde{\alpha}$. Indeed, we have proved in Proposition 1.3.1 that

$$\widetilde{\alpha}(t) = \frac{f(t)}{G(t)} = -\frac{G'(t)}{G(t)}\,.$$

Solving this ODE with initial condition $G(0) = 1$ leads to

$$G(t) = \mathbb{Q}(\tau > t) = \exp\left(-\int_0^t \widetilde{\alpha}(u)\,du\right).$$

Note that $\widetilde{\alpha}(t)dt = \mathbb{Q}(\tau \in dt | \tau > t)$: this is the probability that τ occurs in the interval $[t, t+dt]$ knowing that τ has not yet occured.

1.3.2 CDS market value

For the sake of notational simplicity, we assume in this section that the interest rate r is null, so that the price of a savings account is $B_t = 1$ for every t. We moreover assume that the contractual spread κ is paid in continuous time (i.e., during the time interval $[t, t+dt]$ the amount κdt is paid by the protection buyer to the protection seller). We also consider that the payment at default time is a deterministic function of the default time, i.e., $\xi(\tau)$, which allows to deal with time dependent recovery rates[6]. Let us remark that the results described below can be easily extended to the case of a constant interest rate r or if cash-flows on the premium leg are more realistic.

We saw in Section 1.1 that the cash-flows of a CDS could be divided in two legs : the default leg and the premium leg. The time-t market value of a buy protection position on a CDS is equal to :

$$V_t(\kappa) = D_t - \kappa \cdot P_t, \tag{1.11}$$

where D_t is the time-t present value of the default leg and P_t is the time-t present value of the premium leg per unit of κ. This corresponds to the amount a buyer of protection is willing to pay (or gain) in order to close his position at time t. Let us recall that the contractual spread κ is such that the CDS market value is equal to zero at inception ($V_0(\kappa) = 0$).

We first focus on price dynamics of a CDS with spread κ initiated at time 0. The time-t market price of a CDS maturing at T with contractual spread κ is then given by the formula

[6] $\xi(\tau)$ is equal to the loss given default associated with the reference entity times the notional of the CDS.

$$V_t(\kappa) = \mathbb{E}\Big(\xi(\tau)\mathbb{1}_{\{t<\tau\le T\}} - \mathbb{1}_{\{t<\tau\}}\kappa\big((\tau\wedge T)-t\big)\,\Big|\,\mathscr{H}_t\Big). \tag{1.12}$$

Proposition 1.3.2 *The price at time $t \in [0,T]$ of a credit default swap with spread κ is*

$$V_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\widetilde{V}_t(\kappa), \quad \forall t \in [0,T],$$

where $\widetilde{V}_t(\kappa)$, a deterministic function, stands for the pre-default value of a CDS and equals

$$\widetilde{V}_t(\kappa) = \frac{1}{G(t)}\left(-\int_t^T \xi(u)\,dG(u) - \kappa\int_t^T G(u)\,du\right).$$

Proof. From Lemma 1.3.1, we have, on the set $\{t<\tau\}$,

$$\begin{aligned} V_t(\kappa) &= -\frac{\int_t^T \xi(u)\,dG(u)}{G(t)} - \kappa\left(\frac{-\int_t^T u\,dG(u) + TG(T)}{G(t)} - t\right) \\ &= \frac{1}{G(t)}\left(-\int_t^T \xi(u)\,dG(u) - \kappa\Big(TG(T) - tG(t) - \int_t^T u\,dG(u)\Big)\right). \end{aligned}$$

where, in the last equality, we have used an integration by parts to obtain

$$\int_t^T G(u)\,du = TG(T) - tG(t) - \int_t^T u\,dG(u).$$

□

1.3.3 CDS market Spreads

Like traditional interest-rate swaps, CDS quotations are based on spreads, though this is likely to be modified after the ISDA big bang protocol. Quoted spreads will be after that only a way to express upfront premiums. Let us consider a CDS initiated at time 0 with maturity T and contractual spread κ. The time-t market spread is defined as the contractual spread of the contract if it would have been initiated at time t. In other words, this is the level of the spread $\kappa = \kappa(t,T)$ that makes a T-maturity CDS worthless at time t. A CDS market spread at time t is thus determined by the equation $V_t(\kappa(t,T)) = 0$ where V_t is defined in Proposition 1.3.2.

The T-maturity market spread $\kappa(t,T)$ is therefore a solution to the equation

$$\int_t^T \xi(u)\,dG(u) + \kappa(t,T)\int_t^T G(u)\,du = 0,$$

and thus for every $t \in [0,T]$,

$$\kappa(t,T) = -\frac{\int_t^T \xi(u)\,dG(u)}{\int_t^T G(u)\,du}. \tag{1.13}$$

There exists liquidly quoted CDS spreads on most big companies, and standard maturities are $T = 3, 5, 7, 10$ years. Given Equation 1.13, it is possible to extract a market-implied survival distribution $G(t) = \mathbb{Q}(\tau > t)$, $t \geq 0$ from the term structure of CDS market spreads. See Chapter 3 of [59] or Chapter 2 and 3 of [19] for more details.

In what follows, we fix the maturity date T, and we write briefly $\kappa(t)$ instead of $\kappa(t,T)$. There is a simple relationship between credit spreads and market values. The market price of a CDS with payment ξ at default, maturity T and contractual spread κ equals, for every $t \in [0,T]$,

$$V_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\,(\kappa(t) - \kappa)\,\frac{\int_t^T G(u)\,du}{G(t)},$$

or more explicitly,

$$V_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\,\frac{\int_t^T G(u)\,du}{G(t)}\left(\frac{\int_0^T \xi(u)\,dG(u)}{\int_0^T G(u)\,du} - \frac{\int_t^T \xi(u)\,dG(u)}{\int_t^T G(u)\,du}\right).$$

The latter expression simply means that the value of a CDS contract for a buyer of protection is positive when the current market spread $\kappa(t)$ is greater than the contractual spread κ.

1.3.4 Dynamics of CDS Prices in a single default setting

Proposition 1.3.3 *The dynamics of the (ex-dividend) price $V_t(\kappa)$ on $[0,T]$ are*

$$dV_t(\kappa) = -V_{t-}(\kappa)\,dM_t + (1-N_t)(\kappa - \xi(t)\widetilde{\alpha}(t))\,dt,$$

where the $(\mathbb{Q},\mathbb{H})$-martingale M is given in Proposition 1.3.1.

Proof. It suffices to recall that

$$V_t(\kappa) = (1-N_t)\widetilde{V}_t(\kappa)$$

with $\widetilde{V}$ given in Proposition 1.3.2, so that, using integration by parts formula,

$$dV_t(\kappa) = (1-N_t)\,d\widetilde{V}_t(\kappa) - \widetilde{V}_{t-}(\kappa)\,dN_t.$$

Using the explicit expression of $\widetilde{V}_t(\kappa)$, we find easily that we have

$$d\widetilde{V}_t(\kappa) = \widetilde{\alpha}(t)\widetilde{V}_t(\kappa)\,dt + (\kappa - \xi(t)\widetilde{\alpha}(t))\,dt.$$

The SDE for $V(\kappa)$ follows. $\square$

It is worthwhile to note that the price dynamics is not a martingale under the risk-neutral probability, despite the fact that the interest rate is null. This is because we are dealing with the ex-dividend price. The premium κ is similar to a dividend to be paid, hence the quantity $\kappa(1-N_t)dt$ appears. The quantity $\xi(t)$ can be interpreted as a dividend to be received, at time t, with probability $\widetilde{\alpha}(t)dt$. At default time, the price jumps from $V_{\tau-}(\kappa)$ to 0, as can be seen in the right-hand side of the dynamics.

1.4 Two default times

Let us now study the case with two random times τ_1, τ_2. We denote by $(N_t^i,\, t \geq 0)$ the default process associated with τ_i, $i = 1,2$. The filtration generated by the process N^i is denoted $\mathbb{H}^i$ and the filtration generated by the two processes N^1, N^2 is $\mathbb{H} = \mathbb{H}^1 \vee \mathbb{H}^2$.

Note that an $\mathscr{H}_t^1 \vee \mathscr{H}_t^2$-measurable random variable is

- a constant on the set $t < \tau_1 \wedge \tau_2$,
- a $\sigma(\tau_1 \wedge \tau_2)$-measurable random variable on the set $\tau_1 \wedge \tau_2 \leq t < \tau_1 \vee \tau_2$, i.e., a $\sigma(\tau_1)$-measurable random variable on the set $\tau_1 \leq t < \tau_2$, and a $\sigma(\tau_2)$-measurable random variable on the set $\tau_2 \leq t < \tau_1$. We recall that a $\sigma(\tau_1)$-measurable random variable is a Borel function of τ_1.
- a $\sigma(\tau_1, \tau_2)$-measurable random variable (i.e., a Borel function $h(\tau_1, \tau_2)$) on the set $\tau_1 \vee \tau_2 \leq t$.

To summarize, for fixed t, any $\mathscr{H}_t^1 \vee \mathscr{H}_t^2$-measurable random variable Z admits a representation as

$$Z = h\mathbb{1}_{t<\tau_1\wedge\tau_2} + h_1(\tau_1)\mathbb{1}_{\tau_1\leq t<\tau_2} + h_2(\tau_2)\mathbb{1}_{\tau_2\leq t<\tau_1} + h(\tau_1,\tau_2)\mathbb{1}_{\tau_1\vee\tau_2\leq t}\,.$$

We denote by $G(t,s) = \mathbb{Q}(\tau_1 > t, \tau_2 > s)$ the survival probability of the pair (τ_1, τ_2) and we assume that this function is twice differentiable. We denote by $\partial_i G$, the partial derivative of G with respect to the i-th variable, $i = 1,2$. The density of the pair (τ_1, τ_2) is denoted by f. Simultaneous defaults are precluded in this framework, i.e., $\mathbb{Q}(\tau_1 = \tau_2) = 0$.

Even if the case of two default times is more involved, closed form expressions for the intensities are available. It is important to take into account that the choice of the filtration is very important. Indeed, in general, an $\mathbb{H}^1$-martingale is not an $\mathbb{H}^1 \vee \mathbb{H}^2$-martingale. We shall illustrate this important fact below.

1.4.1 Intensities

We present the computation of martingales associated with default times τ_i, $i = 1,2$, in different filtrations. In particular, we shall obtain the computation of the intensities in various filtrations.

- **Filtration** $\mathbb{H}^i$: We study, for any fixed i, the Doob-Meyer decomposition of the submartingale N^i in the filtration $\mathbb{H}^i$. From Proposition 1.3.1, the process

$$N_t^i - \int_0^{t\wedge\tau_i} \frac{f_i(s)}{G_i(s)} ds \tag{1.14}$$

is an $\mathbb{H}^i$-martingale. Here, $1 - G_i(s) = F_i(s) = \mathbb{Q}(\tau_i \le s) = \int_0^s f_i(u)du$. In other terms, the process $(1 - N_t^i)\frac{f_i(t)}{G_i(t)}$ is the $\mathbb{H}^i$-intensity of τ^i.

- **Filtration** $\mathbb{H}$: We recall a general result which allows to compute the intensities of a default time (see [27]).

Lemma 1.4.1 *Let $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F}$ is a reference filtration and $\mathscr{H}_t = \sigma(\tau \wedge t)$ where τ is a random time. Assume that the supermartingale $G_t := \mathbb{P}(\tau > t|\mathscr{F}_t)$ admits the Doob-Meyer decomposition $G_t = Z_t - A_t$ where Z is a martingale and A is a predictable increasing process absolutely continuous with respect to Lebesgue measure. Then*

$$M_t = N_t - \int_0^{t\wedge\tau} \frac{dA_s}{G_s}$$

is a $\mathbb{G}$-martingale.

Proof. The proof relies on the computation of $\mathbb{E}(M_t - M_s|\mathscr{H}_s)$ for $t > s$. See [27] for details. □

In order to find the intensity of τ_1 in a general two defaults setting, we apply the previous lemma to the case $\mathbb{F} = \mathbb{H}^2$ and $\mathbb{H} = \mathbb{H}^1$. The first step is to compute the associated supermartingale (under the risk-neutral probability $\mathbb{Q}$).

Lemma 1.4.2 *The $\mathbb{H}^2$- supermartingale $\mathbb{Q}(\tau_1 > t|\mathscr{H}_t^2)$ equals*

$$G_t^{1|2} := \mathbb{Q}(\tau_1 > t|\mathscr{H}_t^2) = N_t^2 h(t,\tau_2) + (1 - N_t^2)\psi(t) \tag{1.15}$$

where $\psi(t) = G(t,t)/G(0,t)$, and $h(t,v) = \frac{\partial_2 G(t,v)}{\partial_2 G(0,v)}$.

Proof. From Proposition 1.3.1,

$$\mathbb{Q}(\tau_1 > t|\mathscr{H}_t^2) = \mathbb{1}_{t<\tau_2} \frac{\mathbb{Q}(\tau_1 > t, \tau_2 > t)}{\mathbb{Q}(\tau_2 > t)} + \mathbb{1}_{\tau_2 \le t}\mathbb{Q}(\tau_1 > t|\tau_2).$$

It is easy to check that

$$\mathbb{Q}(\tau_1 > t|\tau_2 = u) = \frac{\mathbb{Q}(\tau_1 > t, \tau_2 \in du)}{\mathbb{Q}(\tau_2 \in du)} = h(t,u)$$

and the result follows. □

Proposition 1.4.1 *Let*

$$a_t = N_t^2 \partial_1 h(t,\tau_2) + (1-N_t^2)\frac{\partial_1 G(t,t)}{G(0,t)}\,.$$

The process M^1 defined as

$$\begin{aligned} M_t^1 &:= N_t^1 + \int_0^{t\wedge\tau_1} \frac{a_s}{G_s^{1|2}} ds \\ &= N_t^1 + \int_0^{t\wedge\tau_1\wedge\tau_2} \frac{\partial_1 G(s,s)}{G(s,s)} ds + \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \frac{\partial_{1,2} G(s,\tau_2)}{\partial_2 G(s,\tau_2)} ds \end{aligned}$$

is an $\mathbb{H}$-martingale.

Proof. The proof relies on some Itô's calculus to obtain the Doob-Meyer decomposition of $\mathbb{Q}(\tau_1 > t|\mathscr{H}_t^2)$ and to prove that $dA_t = a_t dt$. We refer the reader to [7] for details. □

This means that the $\mathbb{H}$-intensity of τ_1 takes into account the knowledge of τ_2 and is equal to the deterministic function $-\frac{\partial_1 G(t,t)}{G(t,t)}$ on the set $t < \tau_2$ and to the random quantity $\varphi(t,\tau_2)$ where $\varphi(t,s) = -\frac{\partial_{1,2}G(t,s)}{\partial_2 G(t,s)}$ on the set $t \geq \tau_2$. In a closed form, the processes $N_t^i - \int_0^t \alpha_s^i ds$, $i = 1,2$, are martingales in the same filtration $\mathbb{H}$, where

$$\begin{aligned} \alpha_t^1 &= (1-N_t^1)\left((1-N_t^2)\frac{-\partial_1 G(t,t)}{G(t,t)} - N_t^2 \frac{\partial_{1,2}G(t,\tau_2)}{\partial_2 G(t,\tau_2)}\right) \\ &= (1-N_t^1)(1-N_t^2)\widetilde{\alpha}^1(t) + (1-N_t^1)N_t^2 \widetilde{\alpha}^{1|2}(t,\tau_2) \\ \alpha_t^2 &= (1-N_t^2)\left((1-N_t^1)\frac{-\partial_2 G(t,t)}{G(t,t)} - N_t^1 \frac{\partial_{1,2}G(\tau_1,t)}{\partial_1 G(\tau_1,t)}\right) \\ &= (1-N_t^1)(1-N_t^2)\widetilde{\alpha}^2(t) + N_t^1(1-N_t^2)\widetilde{\alpha}^{2|1}(\tau_1,t) \end{aligned}$$

where

$$\widetilde{\alpha}^i(t) = -\frac{\partial_i G(t,t)}{G(t,t)} \tag{1.16}$$

$$\widetilde{\alpha}^{1|2}(t,s) = -\frac{\partial_{1,2}G(t,s)}{\partial_2 G(t,s)}, \quad \widetilde{\alpha}^{2|1}(s,t) = -\frac{\partial_{1,2}G(s,t)}{\partial_1 G(s,t)}. \tag{1.17}$$

Note that the minus signs in the value of the intensity are due to the fact that G is decreasing with respect to its component, hence the first order derivatives are non-positive and the second order derivative $\partial_1\partial_2 G$ – equal to the density of the pair (τ_1,τ_2) – is non-negative. The quantity $\widetilde{\alpha}^1(t)dt$ is equal to $\mathbb{Q}(\tau_1 \in dt|\tau_1 \wedge \tau_2 > t)$.

The quantity $\widetilde{\alpha}^{1|2}(t,s) = -\frac{f(t,s)}{\partial_2 G(t,s)}$ evaluated at $s = \tau_2$, represents the value of the predefault intensity process of τ_1 with respect to the filtration $\mathbb{H}$ on the event $\{\tau_2 < t\}$.

Let us remark that, in the particular case where τ_1 and τ_2 are independent (or if $\tau_1 < \tau_2$), the $\mathbb{H}$ intensity of τ_1 equals its $\mathbb{H}^1$ intensity.

This model is very general. Let us note that it is not a Markov model, except if $h(t,s)$ does not depend on s (see [8] for a formal proof). Moreover, it can be extended at the price of notational complexity to n names but computations are not so easy, since they involve partial derivatives of the joint survival function and do not usually lead to tractable Markov processes.

Since we are working in the same filtration[7] the compensated martingale of the counting process $N_t = N_t^1 + N_t^2 = \sum_{i=1}^2 \mathbb{1}_{\tau_i \le t}$ is $M_t := N_t - \int_0^t \alpha_s ds$ where

$$\begin{aligned}
\alpha_t &= \alpha_t^1 + \alpha_t^2 \\
&= (1-N_t^1)(1-N_t^2)\left(-\frac{\partial_1 G(t,t) + \partial_2 G(t,t)}{G(t,t)}\right) \\
&- (1-N_t^1)N_t^2 \frac{\partial_{1,2} G(t,\tau_2)}{\partial_2 G(t,\tau_2)} - (1-N_t^2)N_t^1 \frac{\partial_{1,2} G(\tau_1,t)}{\partial_2 G(\tau_1,t)}.
\end{aligned}$$

It is proved in Bielecki et al. [6] that the process N is Markov if and only if the quantities $\frac{\partial_{1,2} G(t,\tau_2)}{\partial_2 G(t,\tau_2)}$ and $\frac{\partial_{1,2} G(\tau_1,t)}{\partial_2 G(\tau_1,t)}$ are deterministic.

1.4.2 Dynamics of CDS prices in a two defaults setting

Let us now examine the valuation of a single-name CDS written on the default τ_1. Our aim is to show that the dynamics of this CDS will be affected by the information on τ_2: when τ_2 occurs, the intensity of τ_1 changes, and this will change the parameters of the price dynamics.

We consider a CDS

- with a continuously paid constant premium κ,
- which delivers $\xi(\tau_1)$ at time τ_1 if $\tau_1 < T$, where ξ is a deterministic function. In the simplest case ξ is constant, corresponding to constant recovery rates. We recall that ξ corresponds to the loss given default times the nominal of the CDS.

The value of the CDS takes the form

$$V_t(\kappa) = \widetilde{V}_t(\kappa)\mathbb{1}_{t<\tau_2\wedge\tau_1} + \widehat{V}_t(\kappa)\mathbb{1}_{\tau_1\wedge\tau_2 \le t<\tau_1}\,.$$

[7] The sum of two martingales in the same filtration is a martingale.

First, we restrict our attention to the case $t < \tau_2 \wedge \tau_1$.

Proposition 1.4.2 *On the set $t < \tau_2 \wedge \tau_1$, the value of the CDS is*

$$\widetilde{V}_t(\kappa) = \frac{1}{G(t,t)} \left(-\int_t^T \xi(u)\partial_1 G(u,t)\,du - \kappa \int_t^T G(u,t)\,du \right) .$$

Proof. The value $V(\kappa)$ of this CDS, computed in the filtration $\mathbb{H}$ including the information related to the second default, is

$$V_t(\kappa) = \mathbb{1}_{t<\tau_1} \mathbb{E}(\xi(\tau_1)\mathbb{1}_{\tau_1 \le T} - \kappa((T \wedge \tau_1) - t)|\mathscr{H}_t) .$$

Let us denote by $\tau = \tau_1 \wedge \tau_2$ the first default time. Then, $\mathbb{1}_{t<\tau} V_t(\kappa) = \mathbb{1}_{t<\tau} \widetilde{V}_t(\kappa)$, where

$$\begin{aligned}
\widetilde{V}_t(\kappa) &= \frac{1}{\mathbb{Q}(\tau > t)} \mathbb{E}(\xi(\tau_1)\mathbb{1}_{\tau_1 \le T}\mathbb{1}_{t<\tau} - \kappa((T \wedge \tau_1) - t)\mathbb{1}_{t<\tau}) \\
&= \frac{1}{G(t,t)} \mathbb{E}(\xi(\tau_1)\mathbb{1}_{\tau_1 \le T}\mathbb{1}_{t<\tau} - \kappa((T \wedge \tau_1) - t)\mathbb{1}_{t<\tau}) \\
&= \frac{1}{G(t,t)} \left(\int_t^T \xi(u)\mathbb{Q}(\tau_1 \in du, \tau_2 > t) \right. \\
&\qquad \left. - \kappa \int_t^T (u-t)\mathbb{Q}(\tau_1 \in du, \tau_2 > t) - (T-t)\kappa \int_T^\infty \mathbb{Q}(\tau_1 \in du, \tau_2 > t) \right) .
\end{aligned}$$

In other terms, using integration by parts formula

$$\widetilde{V}_t(\kappa) = \frac{1}{G(t,t)} \left(-\int_t^T \xi(u)\partial_1 G(u,t)\,du - \kappa \int_t^T G(u,t)\,du \right) .$$

□

On the event $\{\tau_2 \le t < \tau_1\}$, the CDS price equals

$$\begin{aligned}
V_t(\kappa) &= \hat{V}_t = \mathbb{1}_{t<\tau_1} \mathbb{E}(\xi(\tau_1)\mathbb{1}_{\tau_1 \le T} - \kappa((T \wedge \tau_1) - t)|\sigma(\tau_2)) \\
&= \frac{1}{\partial_2 G(t,\tau_2)} \left(-\int_t^T \xi(u) f(u,\tau_2)\,du - \kappa \int_t^T \partial_2 G(u,\tau_2)\,du \right) := V_t^{1|2}(\tau_2)
\end{aligned}$$

where

$$V_t^{1|2}(s) = \frac{1}{\partial_2 G(t,s)} \left(-\int_t^T \xi(u) f(u,s)\,du - \kappa \int_t^T \partial_2 G(u,s)\,du \right) .$$

In the financial interpretation, $V_t^{1|2}(s)$ is the market price at time t of a CDS on the first credit name, under the assumption that the default τ_2 occurred at time s and the first name has not yet defaulted (recall that simultaneous defaults are excluded, since we have assumed that G is differentiable).

Differentiating the deterministic function which gives the value of the CDS leads to the following result:

Proposition 1.4.3 *The price of a CDS is* $V_t(\kappa) = \widetilde{V}_t(\kappa)\mathbb{1}_{t<\tau_2\wedge\tau_1} + \hat{V}_t(\kappa)\mathbb{1}_{\tau_2\wedge\tau_1\le t<\tau_1}$. *The dynamics of* $\widetilde{V}(\kappa)$ *are*

$$d\widetilde{V}_t(\kappa) = \Big(\big(\widetilde{\alpha}_1(t)+\widetilde{\alpha}_2(t)\big)\widetilde{V}_t(\kappa) + \kappa - \widetilde{\alpha}_1(t)\xi(t) - \widetilde{\alpha}_2(t)V_t^{1|2}(t)\Big)\,dt,$$

where for $i=1,2$ *the function* $\widetilde{\alpha}_i(t)$ *is the (deterministic) pre-default intensity of* τ_i *given in (1.16). The dynamics of* $\widehat{V}(\kappa)$ *are*

$$d\widehat{V}_t(\kappa) = \Big(\widetilde{\alpha}^{1|2}(t,\tau_2)\Big(\widehat{V}_t(\kappa) - \xi(t)\Big) + \kappa\Big)\,dt$$

where $\widetilde{\alpha}^{1|2}(t,s)$ *is given in (1.17).*

Hence, differentiating $V_t = \widetilde{V}_t(1-N_t^1)(1-N_t^2) + \hat{V}_t(1-N_t^1)N_t^2$ one obtains

$$\begin{aligned} dV_t &= (1-N_t^1)(1-N_t^2)d\widetilde{V}_t + (1-N_t^1)N_t^2 d\widehat{V}_t - V_{t-}dN_t^1 \\ &\quad + (1-N_t^1)(V_t^{1|2}(t) - \widetilde{V}_t)dN_t^2 \end{aligned}$$

which leads after light computations[8] to

$$\begin{aligned} dV_t &= (1-N_t^1)(1-N_t^2)(\kappa - \xi(t)\widetilde{\alpha}^1(t))dt + (1-N_t^1)N_t^2(\kappa - \xi(t)\widetilde{\alpha}^{1|2}(t,\tau_2))dt \\ &\quad - V_{t-}dM_t^1 + (1-N_t^1)(V_t^{1|2}(t) - \widetilde{V}_t)dM_t^2 \qquad (1.18) \\ &= \text{dividend part} - V_{t-}dM_t^1 + (1-N_t^1)(V_t^{1|2}(t) - \widetilde{V}_t)dM_t^2\,. \end{aligned}$$

Assume now that a CDS written on τ_2 is also traded in the market. We denote by $V^i, i=1,2$ the prices of the two CDS. Since the CDS are paying premiums, a self financing strategy consisting in ϑ^i units of CDS's has value $X_t = \vartheta_t^1 V_t^1 + \vartheta_t^2 V_t^2$ and dynamics

$$\begin{aligned} dX_t &= \vartheta_t^1\Big(-V_{t-}^1 dM_t^1 + (1-N_t^1)(V_t^{1|2}(t) - \widetilde{V}_t^1)dM_t^2\Big) \\ &\quad + \vartheta_t^2\Big(-V_{t-}^2 dM_t^2 + (1-N_t^2)(V_t^{2|1}(t) - \widetilde{V}_t^2)dM_t^1\Big) \\ &= \Big(-\vartheta_t^1 V_{t-}^1 + \vartheta_t^2(1-N_t^2)(V_t^{2|1}(t) - \widetilde{V}_t^2\Big)dM_t^1 \\ &\quad + \Big(\vartheta_t^1(1-N_t^1)(V_t^{1|2}(t) - \widetilde{V}_t^1) - \vartheta_t^2 V_{t-}^2\Big)dM_t^2\,. \end{aligned}$$

In order to duplicate a claim with value

---

[8] From the definition, one has $dV_t = (1-N_t^1)(1-N_t^2)\cdots + (1-N_t^1)(\widehat{V}_t(\tau_2) - \widetilde{V}_t)dN_t^2$. It is important to note that $\widehat{V}_t(\tau_2)dN_t^2 = V_t^{1|2}(t)dN_t^2$: a computation using $\widehat{V}_t(\tau_2)dN_t^2 = \widehat{V}_t(\tau_2)(dM_t^2 + \ldots dt)$ would lead to a quantity $\widehat{V}_t(\tau_2)dM_t^2$ which has a meaning, but which is NOT a martingale, due to the lack of adapteness of the coefficient $\widehat{V}_t(\tau_2)$.

$$A_t = \mathbb{E}(A) + \int_0^t \delta_t^1 dM_t^1 + \int_0^t \delta_t^2 dM_t^2$$

it remains to solve the linear system

$$\begin{cases} -\vartheta_t^1 V_{t-}^1 + \vartheta_t^2 (1 - N_t^2)(V_t^{2|1}(t) - \widetilde{V}_t^2) = \delta_t^1 , \\ \vartheta_t^1 (1 - N_t^1)(V_t^{1|2}(t) - \widetilde{V}_t^1) - \vartheta_t^2 V_{t-}^2 \quad = \delta_t^2 . \end{cases}$$

Thus, under standard invertibility conditions, one can easily use actually traded CDS instead of instantaneous digital CDS when replicating the claim A.

Chapter 2
Hedging Default Risks of CDOs in Markovian Contagion Models

Areski Cousin, Monique Jeanblanc, Jean-Paul Laurent

When dealing with CDO tranches, the market approach to the derivation of credit default swap deltas consists in bumping the credit curves of the names and computing the ratios of changes in present value of the CDO tranches and the hedging credit default swaps. This involves a pricing engine for CDO tranches, usually some mixture of copula and base correlation approaches, leading to some "market delta".

The only rationale of this modus operandi is local hedging with respect to credit spread risks, provided that the trading books are marked-to-market with the same pricing engine. Even when dealing with small changes in credit spreads, there is no guarantee that this would lead to appropriate hedging strategies, especially to cover large spread widenings and possibly defaults. Also, one could think of changes in base correlation correlated with changes in credit spreads.

A number of CDO hedging anomalies in the base correlation approach are reported in [52]. Moreover, the standard approach is not associated with a replicating theory, thus inducing the possibility of unexplained drifts and time decay effects in the present value of hedged portfolios (see [55]).

Unfortunately, the trading desks cannot rely on a sound theory to determine replicating prices of CDO tranches. This is partly due to the dimensionality issue, partly to the stacking of credit spread and default risks. Laurent (2006) [45] considers the case of multivariate intensities in a conditionally independent framework and shows that for large portfolios where default risks are well diversified, one can concentrate on the hedging of credit spread risks and control the hedging errors. In this approach, the key assumption is the absence of contagion effects which implies that credit spreads of survival names do not jump at default times, or equivalently that defaults are not informative. Whether one should rely on this assumption is to be considered with caution, as discussed in the empirical studies [4] and [16][1]. Moreover, anecdotal evidence such as the failures of Delphi, Enron, Parmalat and WorldCom shows mixed results.

In this chapter, we adopt the framework of Laurent, Cousin and Fermanian (2007) [46], concentrating on default risks, credit spreads and dependence dynamics being driven by the arrival of defaults. We will calculate so-called "credit deltas", that are the present value impacts of some default event on a given CDO tranche, divided by the present value impact of the hedging instrument (here the underlying index) under the same scenario. Contagion models were introduced to the credit field by Davis and Lo (2001) [17], Jarrow and Yu (2001) [38] and further studied by Yu (2007) [65]. Schönbucher and Schubert (2001) [61] show that copula models exhibit some contagion effects and relate jumps of credit spreads at default times to the partial derivatives of the copula. This is also the framework used by Bielecki, Jeanblanc and Rutkowski (2007) [7] to address the hedging issue. We refer to Section 1.4 of this contribution for a detailed discussion of this topic. A similar but somehow more tractable approach has been considered by Frey and Backhaus (2007) [31], since the latter paper considers some Markovian models of contagion. In a copula model, the contagion effects are computed from the dependence structure of default times, while in contagion models the intensity dynamics are the inputs from which the dependence structure of default times is derived. In both approaches, credit spreads shifts occur only at default times. Thanks to this quite simplistic assumption, and provided that no simultaneous defaults occurs, it can be shown that the CDO market is complete, i.e., CDO tranche cash-flows can be fully replicated by dynamically trading individual credit default swaps or, in some cases, by trading the credit default swap index (see Subsection 1.2.3 of this contribution for a presentation of the theoretical ideas).

In this chapter we focus on the hedging of synthetic CDO tranches. For the chapter to be self-contained, we briefly describe in Section 2.1 the cash-flows of a synthetic CDO tranche. While the use of the representation Theorem 1.1 guarantees that, in our framework, any basket default swap can be perfectly hedged with respect to default risks, it does not provide a practical way of constructing hedging strategies. In Section 2.2, we restrict ourselves to the case of homogeneous portfolios with Markovian intensities which results in a dramatic dimensionality reduction for the (risk-neutral) valuation of CDO tranches and the hedging of such tranches as well. We find out that the aggregate loss is associated with a pure birth process, which is now well documented in the credit literature. Section 2.3 provides an overview of the calibration methods proposed in the literature on contagion credit risk models. We investigate in particular a calibration method based on the marginal distributions of the number of defaults. Section 2.4 details the computation of replicating strategies of CDO tranches with respect to the credit default swap index, through

[1] The conclusions of this paper have been disputed by [44] in which the conditional independence assumption have not be rejected when tested on the same default database. These discrepancies are explained by an alternative specification of individual default intensities.

a recombining tree on the aggregate loss. We discuss how hedging strategies are related to dependence assumptions in Gaussian copula and base correlation frameworks. We also compare the replicating strategies obtained in the contagion model with the hedging ratios (spread sensitivity ratios) provided by the Gaussian copula approach or computed in alternative credit risk models.

2.1 Synthetic CDO tranches

Synthetic CDOs are structured products based on an underlying portfolio of reference entities subject to credit risk. It allows investors to sell protection on specific risky portion or tranche of the underlying credit portfolio depending on their desired risk-profile. A synthetic CDO structure is initially arranged by a financial institution (typically an investment bank) which holds a credit portfolio composed of CDS (see figure 2.1). This CDS portfolio is then transferred to a subsidiary company commonly called a special purpose vehicle (SPV). The SPV redistributes the credit risk of the underlying portfolio by raising specific credit-protection products corresponding to different levels of risk. The SPV liability side is defined by the different tranches that have been sold and the asset side corresponds to the portfolio of CDS. The incomes generated by the pool of CDS (premium payments) are re-allocated to the different tranches using a precise prioritization scheme. An investor (seller of protection) on a CDO tranche receives a higher premium if the tranche has a lower level of subordination. For example, the equity tranche which covers the fist losses on the underlying portfolio receives the highest income.

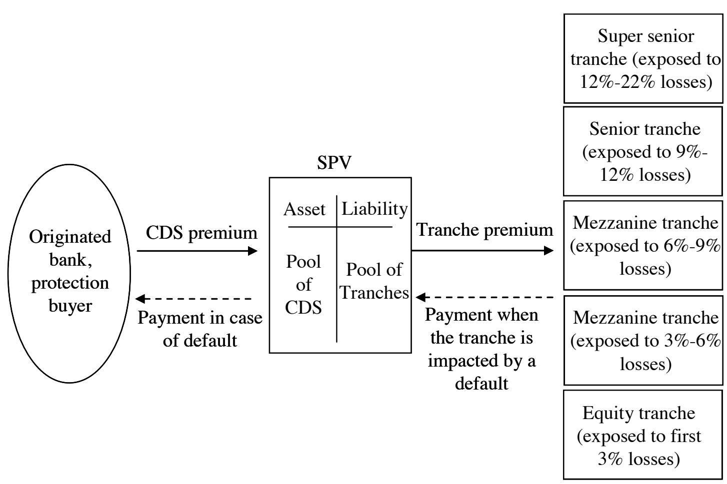

Fig. 2.1 Structure of a synthetic CDO

2.1.1 Credit Default Swap Indices

A Credit Default Swap Index (CDS Index) is a multi-name credit derivative which allows market participants to buy and sell protection directly on a pool of Credit Default Swaps.

CDS indices are actively traded. This means that it can be easier to hedge a credit derivatives referencing a portfolio of CDS with an index than it would be to buy many CDS to achieve a similar effect. This is the reason why a popular use of CDS indices is to hedge multi-name credit positions.

There are currently two main families of CDS indices: CDX and iTraxx. CDX indices contain North American and Emerging Market companies and iTraxx contain companies from the rest of the world (mainly Europe and Asia). The iTraxx Europe Main and the CDX North America Main are the most liquid CDS indices. Each Main index includes 125 equally weighted CDS issuers from their respective region[2]. These issuers are investment grade at the time an index series is launched, with a new series launched every six months. In practice, “on the run” Main indices are mostly composed of A-rated and BBB-rated issuers.

2.1.2 Standardized CDO tranches

Market-makers of these indices have also agreed to quote standard tranches on these portfolios from equity or first loss tranches to the most senior tranches.

Each tranche is defined by its attachment point which defines the level of subordination and its detachment point which defines the maximum loss of the underlying portfolio that would result in a full loss of tranche notional. The first-loss 0-3% equity tranche is exposed to the first several defaults in the underlying portfolio. This tranche is the riskiest as there is no benefit of subordination but it also offers high returns if no defaults occur. The junior mezzanine 3-6% and the senior mezzanine 6-9% tranches are less immediately exposed to the portfolio defaults but the premium received by the protection seller is smaller than for the equity tranche. The 9-12% tranche is the senior tranche, while the 12-20% tranche is the low-risk super senior piece. As illustrated in figure 2.2 and figure 2.3, the tranching of the indices in Europe and North America is different. In North America, the CDX index is tranched into standard classes representing equity 0-3%, junior mezzanine 3-7%, senior mezzanine 7-10%, senior 10-15% and super senior 15-30% tranche.

[2] The proportion of each issuer in the pool is equal to $1/125 = 0.08\%$

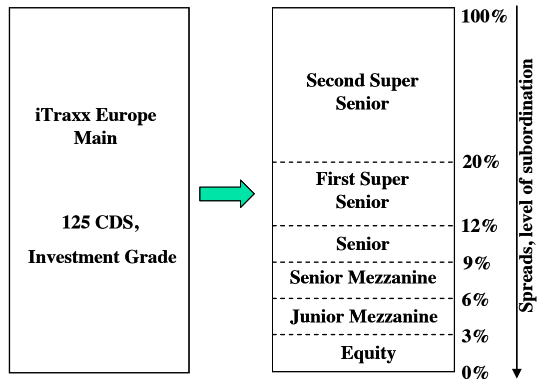

Fig. 2.2 Standardized CDO tranches on iTraxx Europe Main.

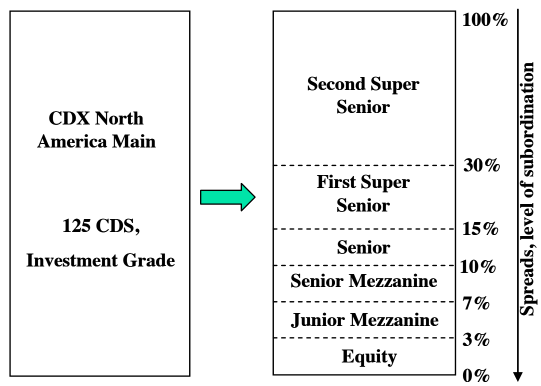

Fig. 2.3 Standardized CDO tranches on CDX North America Main.

For a detailed description of the credit derivatives market, the reader is referred to the textbooks [15], [19], [39], [49] or [59]. Before addressing the hedging issue of CDO tranches, let us describe the cash-flows associated with these products.

2.1.3 Cash-flows description

We adopt the same notation as in Chapter 1 and we work under the risk-neutral probability $\mathbb{Q}$ defined in Subsection 1.2.3. We consider a portfolio of n credit references and we denote by $(\tau_1,\ldots,\tau_n)$ the random vector of default times defined on the probability space $(\Omega,\mathscr{G},\mathbb{Q})$. If name i defaults, it drives a loss of $\xi_i = E_i\,(1-R_i)$ where E_i denotes the nominal amount and R_i the recovery rate. The loss given default ξ_i is assumed here to be constant over time. The key quantity for the pricing of synthetic CDO tranches is the cumulative loss

$$L_t = \sum_{i=1}^{n} \xi_i N_t^i,$$

where $N_t^i = 1_{\{\tau_i \le t\}}$, $i = 1, \ldots, n$ are the default indicator processes associated with default time τ_i, $i = 1, \ldots, n$. Let us recall that the processes N^i, $i = 1, \ldots, n$ are adapted to the global filtration $\mathbb{H} = (\mathscr{H}_t, t \ge 0)$ where $\mathscr{H}_t = \vee_{i=1}^n \mathscr{H}_t^i$ and $\mathscr{H}_t^i = \sigma(N_s^i, s \le t)$. Let us remark at this stage that the loss process L is an increasing right-continuous pure jump process.

The cash-flows associated with a synthetic CDO tranche only depend upon the realized path of the cumulative loss process L. Default losses on the reference portfolio are split along some thresholds (attachment and detachment points) and allocated to the various tranches. A synthetic CDO tranche can be viewed as a bilateral contract between a protection seller and a protection buyer. In what follows, we consider a synthetic CDO tranche with attachment point a, detachment point b and maturity T and we describe the cash-flows associated with the default payment leg (payments received by the protection buyer) and the cash-flows associated with the premium payment leg (payments received by the protection seller).

Default payments leg

The protection seller agrees to pay the protection buyer default losses each time they impact the tranche (a,b) of the reference portfolio. More precisely, the cumulative default payment $L_t^{(a,b)}$ on the tranche $[a,b]$ is equal to zero if $L_t \le a$, to $L_t - a$ if $a \le L_t \le b$ and to $b - a$ if $L_t \ge b$. Let us remark that $L_t^{(a,b)}$ has a call spread payoff with respect to L_t and can be expressed as $L_t^{(a,b)} = (L_t - a)^+ - (L_t - b)^+$. Default payments are simply equal to the increment of $L_t^{(a,b)}$, i.e., there is a payment of $L_t^{(a,b)} - L_{t-}^{(a,b)}$ from the protection seller at every time a jump of $L_t^{(a,b)}$ occurs before contract maturity T. Figure 2.4 shows a realized path of the loss process $(L_t, t \ge 0)$ and the corresponding path of losses affecting CDO tranche $[a,b]$.

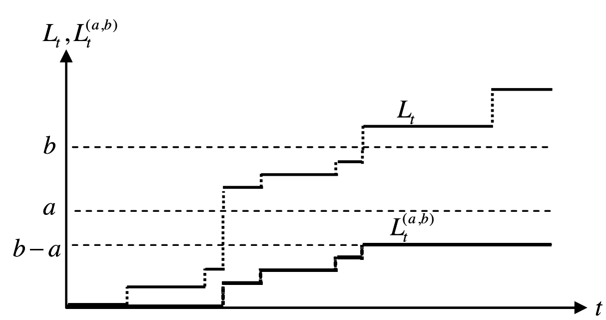

Fig. 2.4 A realized path of the reference portfolio losses and the corresponding path of losses affecting CDO tranche $[a,b]$. Jumps occur at default times

For simplicity we assume that the continuously compounded default free interest rate r_t is deterministic and we denote $B_t(t') = \exp\left(-\int_t^{t'} r_s ds\right)$ the time-t discount factor up to time t' ($t \leq t'$). At time t, the discounted payoff corresponding to default payments after time t can written as:

$$\int_t^T B_t(s) dL_s^{(a,b)} := \sum_{i=1}^n B_t(\tau_i)\left(L_{\tau_i}^{(a,b)} - L_{\tau_i-}^{(a,b)}\right) 1_{\{t<\tau_i \leq T\}}. \tag{2.1}$$

Thanks to Stieltjes integration by parts formula and Fubini theorem, the time-t price of the default payment leg under the risk-neutral measure can be expressed as:

$$\begin{aligned} D_t &= \mathbb{E}\left[\int_t^T B_t(s) dL_s^{(a,b)} \mid \mathscr{H}_t\right] \\ &= B_t(T)\mathbb{E}\left[L_T^{(a,b)} \mid \mathscr{H}_t\right] - L_t^{(a,b)} + \int_t^T r_s B_t(s) \mathbb{E}\left[L_s^{(a,b)} \mid \mathscr{H}_t\right] ds. \end{aligned}$$

Premium payments leg

The protection buyer has to pay the protection seller a periodic premium payment (quarterly for standardized indexes) based on a fixed contractual spread κ and proportional to the current outstanding nominal of the tranche $b - a - L_t^{(a,b)}$. Let us denote by $T_1 < \ldots < T_p$, the premium payment dates with $T_p = T$ and by Δ_i the length of the i-th period $[T_{i-1}, T_i]$ (in fractions of a year and with convention $T_0 = 0$). The CDO premium payments are equal to $\kappa\Delta_i\left(b - a - L_{T_i}^{(a,b)}\right)$ at regular payment dates T_i, $i = 1, \ldots, p$. Moreover, when a default occurs between two premium payment dates and when it affects the tranche, an additional payment (the accrued coupon) must be made at default time to compensate the change in value of the tranche outstanding nominal. For example, if name j defaults between T_{i-1} and T_i, the associated accrued coupon is equal to $\kappa(\tau_j - T_{i-1})\left(L_{\tau_j}^{(a,b)} - L_{\tau_j-}^{(a,b)}\right)$. Eventually, at time t, the discounted payoff corresponding to premium payments can be expressed as:

$$\sum_{i=p_t}^{p}\left(B_t(T_i)\kappa\Delta_i\left(b - a - L_{T_i}^{(a,b)}\right) + \int_{T_{i-1}}^{T_i} B_t(s)\kappa(s - T_{i-1}) dL_s^{(a,b)}\right), \tag{2.2}$$

where $p_t = \inf\{i = 1, \ldots, p \mid T_i > t\}$ is the index of the first premium payment date after time t and $T_{p_t-1} = t$ by convention.

Using the same computational method as for the default leg, the time-t present value of the premium leg under the risk-neutral measure, denoted P_t, can be expressed as:

$$P_t = \kappa \cdot P_t^u, \tag{2.3}$$

with

$$P_t^u = \sum_{i=p_t}^{p} \left(B_t(T_i)\Delta_i \left(b - a - \mathbb{E}\left[L_{T_i}^{(a,b)} \mid \mathscr{H}_t \right] \right) + AC_{i,t} \right), \tag{2.4}$$

and where

$$AC_{i,t} = B_t(T_i)\Delta_i \mathbb{E}\left[L_{T_i}^{(a,b)} \mid \mathscr{H}_t \right] - \int_{T_{i-1}}^{T_i} B_t(s)\left(1 - r_s\left(s - T_{i-1}\right)\right) \mathbb{E}\left[L_s^{(a,b)} \mid \mathscr{H}_t \right] ds. \tag{2.5}$$

The quantity P_t^u corresponds to the time-t present value of the unitary premium leg (contractual spread κ equal to 1bp).

The CDO tranche (contractual) spread κ is chosen such that the contract is fair at inception, i.e., such that the default payment leg is equal to the premium payment leg :

$$\kappa = \frac{D_0}{P_0^u}.$$

The spread κ is quoted in basis point per annum[3]. Let us remark that the computation of κ only involves the expected losses on the tranche, $\mathbb{E}\left[L_t^{(a,b)}\right]$ at different time horizons. These can readily be derived from the marginal distributions of the aggregate loss on the reference portfolio.

2.1.4 CDO tranche price and market spread

The time-t price (buy protection position) of a CDO tranche (a,b) is such that $V_t(\kappa) = D_t - \kappa \cdot P_t^u$. This corresponds to the amount a buyer of protection is willing to pay (or gain) in order to close his position at time t. Let us note that this is consistent with the definition of the contractual spread κ for which the market value must be equal to zero at inception ($V_0(\kappa) = 0$).

Like CDS, most CDO tranche quotations are based on spreads. The time-t market spread is defined as the contractual spread of a tranche with the same characteristic but initiated at time t:

$$\kappa_t = \frac{D_t}{P_t^u}.$$

Let us also note that there is a simple relationship between credit spreads and market values:

$$V_t(\kappa) = P_t^u\left(\kappa_t - \kappa\right).$$

[3] Let us remark that market conventions are different for the pricing of equity tranches (CDO tranches $(0,b)$ with $0 < b \leq 1$). Due to the high level of risk embedded in these "first losses tranches", the premium κ is fixed beforehand at 500 bps per annum and the protection seller receives an additional payment at inception based on an "upfront premium" and proportional to the size of the tranche. This "upfront premium" is quoted in percentage of the nominal value.

The latter expression simply means that the value of a CDO tranche contract for a buyer of protection is positive when the current market spread is greater than the contractual spread.

As illustrated in Subsection 2.1.2, there exists liquidly quoted CDO tranches on most CDS indices. Figure 2.5 shows the dynamics of credit spreads on the five year iTraxx index between May and November 2007[4]. It is interesting to observe a sharp bump corresponding to the summer 2007 credit crisis.

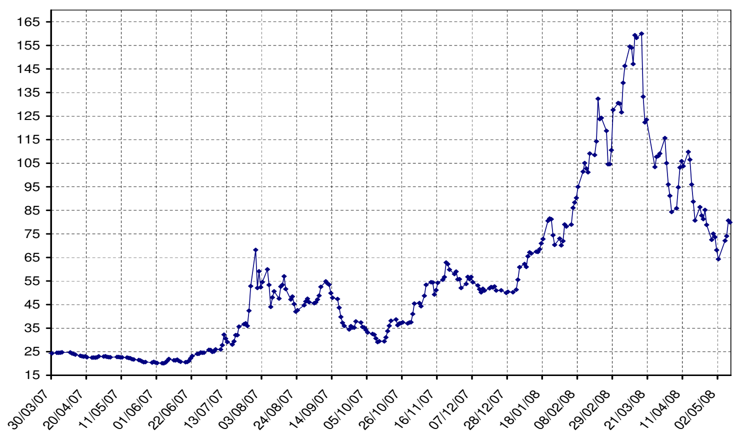

Fig. 2.5 Credit spreads on the five years iTraxx index (Series 7, 8 and 9) in bps, source Markit.

2.2 Homogeneous Markovian contagion models

While the use of the representation Theorem 1.1 guarantees that, in our framework, any basket default swap can be perfectly hedged with respect to default risks, it does not provide a practical way of constructing hedging strategies. As is the case with interest rate or equity derivatives, exhibiting hedging strategies involves some Markovian assumptions.

[4] Apart from details regarding the premium leg, cash-flows generated by a CDS index can be considered to be the same as the ones of a $[0, 100\%]$ CDO tranche

2.2.1 Intensity specification

In the contagion approach, one starts from a specification of the risk-neutral predefault intensities[5] $\tilde{\alpha}^1, \ldots, \tilde{\alpha}^n$. In Chapter 1, risk-neutral predefault intensities depend upon the complete history of defaults. More simplistically, it is often assumed that they depend only upon the current credit status, i.e., the default indicators; thus $\tilde{\alpha}_t^i$, $i = 1, \ldots, n$ are deterministic functions of $N_t^1, \ldots, N_t^n$. In this paper, we will further remain in this Markovian framework, i.e., the default intensities will take the form

$$\tilde{\alpha}_t^i = \tilde{\alpha}^i \left(t, N_t^1, \ldots, N_t^n\right), \ i = 1, \ldots, n. \tag{2.6}$$

This Markovian assumption may be questionable, since the contagion effect of a default event may vanish as time goes by. The Hawkes process, that was used in the credit field by Giesecke and Goldberg (2006) (see [29] or [34]), provides such an example of a more complex time dependence.

Other specifications with the same aim are discussed in [48]. Popular examples are the models of [38], [42], [65], where the intensities are affine functions of the default indicators.

The connection between contagion models and Markov chains is described in the book of Lando [43]. More recently, Herbertsson and Rootzén [37] proved that default times with default intensities defined by Equation 2.6 could be represented as the times until absorption in a finite state absorbing Markov chain. According to Assaf et al. [3] terminology, default times follow a multivariate-phase type distribution in this framework.

Another practical issue is related to name heterogeneity. Modelling all possible interactions amongst names leads to a huge number of contagion parameters and high-dimensional problems, thus to numerical issues. For this practical purpose, we will further restrict to models where all the names share the same risk-neutral intensity[6]. This can be viewed as a reasonable assumption for CDO tranches on large indices, although this is an issue with equity tranches for which idiosyncratic risk is an important feature. Since risk-neutral predefault intensities, $\tilde{\alpha}^1, \ldots, \tilde{\alpha}^n$ are equal, we will further denote these individual predefault intensities by $\tilde{\alpha}$.

For further tractability, we will further rely on a strong name homogeneity assumption, that individual predefault intensities only depend upon the number of defaults. Let us denote by $N_t = \sum_{i=1}^{n} N_t^i$ the number of defaults at time t within the

[5] Let us recall that the default intensity of name i vanishes after τ_i, i.e., $\alpha_t^i = 0$ on the set $\{t > \tau_i\}$.

[6] This means that the predefault intensities have the same functional dependence to the default indicators.

pool of assets. Predefault intensities thus take the form[7] $\tilde{\alpha}_t^i = \tilde{\alpha}(t, N_t)$. This is related to mean-field approaches (see [31]). As for parametric specifications, we can think of some additive effects, i.e. the predefault name intensities take the form $\tilde{\alpha}(t, N_t) = a + b \times N_t$ for some constants a, b as mentioned in [31], corresponding to the "linear counterparty risk model", or multiplicative effects in the spirit of Davis and Lo (2001) [17], i.e., the predefault intensities take the form $\tilde{\alpha}(t, N_t) = a \times b^{N_t}$. Of course, we could think of a non-parametric model. We provide a calibration procedure of such unconstrained intensities onto market inputs in Section 2.3.

For simplicity, we will further assume a constant recovery rate equal to R and a constant exposure among the underlying names. The aggregate fractional loss at time t is given by: $L_t = (1 - R)\frac{N_t}{n}$. As a consequence of the no simultaneous defaults assumption, the intensity of L_t or of N_t is simply the sum of the individual default intensities and is itself only a function of the number of defaults process. Let us denote by $\lambda(t, N_t)$ the risk-neutral loss intensity. It is related to the individual predefault intensities by:

$$\lambda(t, N_t) = (n - N_t) \times \tilde{\alpha}(t, N_t).$$

We are thus typically in a bottom-up approach, where one starts with the specification of name intensities and thus derives the dynamics of the aggregate loss.

2.2.2 *Risk-neutral pricing*

Let us remark that in a Markovian homogeneous contagion model, the process N_t is a continuous time Markov chain (under the risk-neutral probability $\mathbb{Q}$), and more precisely a pure birth process, according to Karlin and Taylor (1975) [40] terminology[8], since only single defaults can occur[9]. The generator of the chain, $\Lambda(t)$ is quite simple:

$$\Lambda(t) = \begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & \cdots & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

---

[7] Let us remark that on $\{\tau_i > t\}$, $N_t = \sum_{j \neq i} N_t^j$, so that the predefault intensity of name i, actually only depends on the credit status of the other names.

[8] According to Feller's terminology, we should speak of a pure death process. Since, we later refer to [40], we prefer their terminology.

[9] Regarding the assumption of no simultaneous defaults, we also refer to [10], [57], [64]. Allowing for multiple defaults could actually ease the calibration to senior CDO tranche quotes.

Such a simple model of the number of defaults dynamics was considered in [60] where it is called the "one-step representation of the loss distribution". The approach described in this chapter can be seen as a bottom-up view of the previous model, where the risk-neutral prices can actually be viewed as replicating prices. As an example of this approach, let us consider the replication price of a European payoff with payment date T, such as a "zero-coupon tranchelet", paying $1_{\{N_T=k\}}$ at time T for some $k \in \{0,1,\ldots,n\}$. Let us denote by $V(t,N_t) = e^{-r(T-t)}\mathbb{Q}(N_T = k|N_t)$ the time-t replication price and by $V(t,.)$ the price vector whose components are $V(t,0), V(t,1),\ldots,V(t,n)$ for $0 \leq t \leq T$. We can thus relate the price vector $V(t,.)$ to the terminal payoff, using the transition matrix $\mathbf{Q}(t,T)$ between dates t and T:

$$V(t,.) = e^{-r(T-t)}\mathbf{Q}(t,T)V(T,.), \tag{2.7}$$

where $V(T,j) = 1_{\{j=k\}},\ j = 0,1,\ldots,n$. The transition matrix solves for the Kolmogorov backward and forward equations :

$$\frac{\partial \mathbf{Q}(t,T)}{\partial t} = -\Lambda(t)\mathbf{Q}(t,T),\ \ \frac{\partial \mathbf{Q}(t,T)}{\partial T} = \mathbf{Q}(t,T)\Lambda(T). \tag{2.8}$$

In the time homogeneous case, i.e., when the generator is a constant $\Lambda(t) = \Lambda$, the transition matrix can be written in exponential form :

$$\mathbf{Q}(t,T) = \exp\left((T-t)\Lambda\right). \tag{2.9}$$

These ideas have been put in practice by [1], [18], [28], [36], [37], [48] and [63]. These papers focus on the pricing of credit derivatives, while our concern here is the feasibility and implementation of replicating strategies.

2.2.3 Computation of credit deltas

We recall that the credit delta with respect to name i is the amount of hedging instruments (the index here, but possibly a i-th credit default swap) that should be bought to be protected against a sudden default of name i. A nice feature of homogeneous contagion models is that the credit deltas are the same for all (the non-defaulted) names, which results in a dramatic dimensionality reduction. In that case, it is enough to consider the index portfolio as a single hedging instrument, which is consistent with some market practices.

Let us consider a European type payoff[10] and denote its replication price at time t by $V(t,.)$. In order to compute the credit deltas, let us remark that, by Itô's lemma,

[10] For notational simplicity, we assume that there are no intermediate payments. This corresponds for instance to the case of zero-coupon CDO tranches with up-front premiums. The more general case is considered in Section 2.4.

$$
\begin{aligned}
dV(t,N_t) &= \frac{\partial V(t,N_{t-})}{\partial t}dt + \left(V(t,N_t) - V(t,N_{t-})\right)dN_t \\
&= \frac{\partial V(t,N_{t-})}{\partial t}dt + \left(V(t,N_{t-}+1) - V(t,N_{t-})\right)dN_t.
\end{aligned}
$$

The second term in the right hand side of the latter expression, $V(t,N_{t-}+1) - V(t,N_{t-})$ is associated with the jump in the price process when a default occurs in the credit portfolio, i.e. $dN_t = 1$. Thanks to the fact that $dN_t = \sum_{i=1}^{n} dN_t^i$ and, since $e^{-rt}V(t,N_t)$ is a $(\mathbb{Q},\mathbb{H})$–martingale, it can be seen using Itô's lemma that V solves for the backward Kolmogorov equations:

$$
\frac{\partial V(t,k)}{\partial t} + \lambda(t,k) \times \left(V(t,k+1) - V(t,k)\right) = rV(t,k),\ k = 0,\ldots,n-1,
$$

where the terminal conditions are given by the payoff function at time T and with $V(t,n) = 0$, for all $t \geq 0$. We end up with:

$$
\begin{aligned}
dV(t,N_t) &= rV(t,N_{t-})\,dt \\
&\quad + \sum_{i=1}^{n} \left(V(t,N_{t-}+1) - V(t,N_{t-})\right)\left(dN_t^i - \tilde{\alpha}(t,N_{t-})(1-N_t^i)dt\right).
\end{aligned}
\tag{2.10}
$$

As a consequence the credit deltas with respect to the individual instantaneous default swaps are equal to:

$$
\delta_t^i = \delta^i(t,N_{t-}) = V(t,N_{t-}+1) - V(t,N_{t-}),
$$

for $0 \leq t \leq T$ and $i = 1,\ldots,n$.

Let us denote by $V^I(t,k) = e^{-r(T-t)}E\left[1 - \frac{N_T}{n}\,|N_t = k\right]$ the time-t price of the equally weighted portfolio involving defaultable bonds and set

$$
\delta_t^I = \delta^I(t,N_{t-}) = \frac{V(t,N_{t-}+1) - V(t,N_{t-})}{V^I(t,N_{t-}+1) - V^I(t,N_{t-})}.
\tag{2.11}
$$

As the dynamics of V^I also satisfies SDE (2.10) and using equation 2.11, we can deduce that:

$$
dV(t,N_t) = r \times \left(V(t,N_{t-}) - \delta^I(t,N_{t-})V^I(t,N_{t-})\right)dt + \delta^I(t,N_{t-})dV^I(t,N_t).
$$

As a consequence, we can perfectly hedge a European type payoff, say a zero-coupon CDO tranche, using only the index portfolio and the risk-free asset. The hedge ratio, with respect to the index portfolio is actually equal to (2.11). The previous hedging strategy is feasible provided that $V^I(t,N_{t-}+1) \neq V^I(t,N_{t-})$. The usual case corresponds to some positive dependence, thus $\alpha(t,0) \leq \alpha(t,1) \leq \cdots \leq$

$\alpha(t,n-1)$. Therefore $V^I(t,N_{t-}+1) < V^I(t,N_{t-})$[11]. The decrease in the index portfolio value is the consequence of a direct default effect (one name defaults) and an indirect effect related to a positive shift in the credit spreads associated with the non-defaulted names.

The idea of building a hedging strategy based on the change in value at default times was introduced in [2]. The rigorous construction of a dynamic hedging strategy in a univariate case can be found in [9]. Our result can be seen as a natural extension to the multivariate case, provided that we deal with Markovian homogeneous models: we simply need to deal with the number of defaults N_t and the index portfolio $V^I(t,N_t)$ instead of a single default indicator N_t^i and the corresponding defaultable discount bond price.

2.3 Calibration of loss intensities

Another nice feature of the homogeneous Markovian contagion model is that the loss dynamics or equivalently the default intensities can be determined from market inputs such as CDO tranche premiums. Since the risk neutral dynamics are unambiguously derived from market inputs, so will be for dynamic hedging strategies of CDO tranches. This greatly facilitates empirical studies, since the replicating figures do not depend upon unobserved and difficult to calibrate parameters.

The construction of the implied Markov chain for the aggregate loss parallels the one made by Dupire (1994) [25] to construct a local volatility model from call option prices. Similar ideas are used in [23], [58] to build up implied trees. Laurent and Leisen (2000) [47] have shown how an implied Markov chain can be derived from a discrete set of option prices. In these approaches, the calibration of the implied dynamics on market inputs involves forward Kolmogorov equations. Starting from a complete set of CDO tranche premiums or equivalently from a complete set of number of default distributions, [60] provided the construction of the loss intensities. Similarities between the Dupire's approach and the building of the one step Markov chain of [60] have also been reported in [14], [20] and [48]. We propose now to detail and comment the latter calibration approach of loss intensities.

[11] In the case where $\alpha(t,0) = \alpha(t,1) = \ldots = \alpha(t,n)$, there are no contagion effects and default dates are independent. We still have $V^I(t,N_{t-}+1) < V^I(t,N_{t-})$ since $V^I(t,N_{t-})$ is linear in the number of surviving names.

2.3.1 Calibration of loss intensities on a complete set of number of defaults probabilities

While the pricing and thus the hedging involves a backward procedure, calibration is associated with forward Kolmogorov differential equations. We show here a non-parametric fitting procedure of a possibly non time homogeneous pure birth process onto a complete set of marginal distributions of number of defaults. This is quite similar to the one described in [60], though the purpose is somehow different since the aim of [60] was to construct arbitrage-free, consistent with some complete loss surface, Markovian models of aggregate losses, possibly in incomplete markets, without detailing the feasibility and implementation of replicating strategies.

We will further denote the marginal default probabilities by $p(t,k) = \mathbb{Q}(N_t = k)$ for $0 \le t \le T$, $k = 0,1,\dots,n$. Clearly, this involves more information that one could directly access through the quotes of liquid CDO tranches, especially with respect to small and large number of defaults. As for the computation of the number of default probabilities from quoted CDO tranche premiums, we refer to [41], [32], [50], [53], [64] and [62]. Practical issues related to the calibration inputs are also discussed in [63].

In the case of a pure birth process, the forward Kolmogorov equations can be written as:

$$\begin{cases} \frac{dp(t,0)}{dt} = -\lambda(t,0)p(t,0), & k = 0, \\ \frac{dp(t,k)}{dt} = \lambda(t,k-1)p(t,k-1) - \lambda(t,k)p(t,k), & k = 1,\dots,n. \end{cases} \tag{2.12}$$

Since the space state is finite, there are no regularity issues and these equations admit a unique solution[12] (see below for practical implementation). These forward equations can be used to compute the loss intensity dynamics $t \in [0,T] \to \lambda(t,N_t)$, thanks to:

$$\begin{cases} \lambda(t,0) = -\frac{1}{p(t,0)}\frac{dp(t,0)}{dt}, & k = 0, \\ \lambda(t,k) = \frac{1}{p(t,k)}\left[\lambda(t,k-1)p(t,k-1) - \frac{dp(t,k)}{dt}\right], & k = 1,\dots,n, \end{cases} \tag{2.13}$$

for $0 \le t \le T$. Let us remark that we can also write:

$$\lambda(t,k) = -\frac{1}{p(t,k)}\frac{d\sum_{m=0}^{k} p(t,m)}{dt} = -\frac{1}{\mathbb{Q}(N_t = k)}\frac{d\mathbb{Q}(N_t \le k)}{dt}, \quad k = 0,\dots,n. \tag{2.14}$$

Eventually, the name intensities are provided by: $\tilde{\alpha}(t,N_t) = \frac{\lambda(t,N_t)}{n-N_t}$. This shows that we can fully recover the loss intensities from the marginal distributions of the number of defaults, if the latters do not occur simultaneously. This has to be related to

[12] We refer to [40] for more details about the forward equations in the case of a pure birth process.

the result of Cont and Minca [14] which states that, under the assumption of no simultaneous defaults, the flow of marginal loss distributions associated with a general point process can be matched with the one of a Markovian jump process.

On practical grounds, the computation of the $p(t,k)$ usually involves some arbitrary smoothing procedure and hazardous extrapolations for small time horizons. For these reasons, we think that it is more appropriate and reasonable to calibrate the Markov chain of aggregate losses on a discrete set of meaningful market inputs corresponding to liquid maturities.

2.3.2 *Calibration of time homogeneous loss intensities*

In practical applications, we can only rely on a discrete set of loss distributions corresponding to liquid CDO tranche maturities. In the examples below, we will calibrate the loss intensities given a single calibration date T. For simplicity, we will be given the default probabilities $p(T,k), k=0,1,\ldots,n$. Now and in the sequel, we assume that the loss intensities are time homogeneous: the intensities do not depend on time but only on the number of realized defaults. We further denote by $\lambda_k = \lambda(t,k)$ for $0 \leq t \leq T$, the loss intensity for $k=0,1,\ldots,n-1$. Let us note that [28] have also developed a similar computation of the loss intensities λ_k from the values of default probabilities.

Solving the forward equations (2.12) provides

$$\begin{cases} p(T,0) = e^{-\lambda_0 T}, & k=0, \\ p(T,k) = \lambda_{k-1} \int_0^T e^{-\lambda_k (T-s)} p(s,k-1) ds, & k=1,\ldots,n-1. \end{cases} \tag{2.15}$$

The previous equations can be used to determine $\lambda_0,\ldots,\lambda_{n-1}$ iteratively, even if our calibration inputs are the defaults probabilities at the single date T.

Assume for the moment that the intensities $\lambda_0,\ldots,\lambda_{n-1}$ are known, positive and distinct[13]. To solve the forward equations, we assume that the default probabilities can be written as

$$p(t,k) = \sum_{i=0}^{k} a_{k,i} e^{-\lambda_i t}, \;\; 0 \leq t \leq T, \; k=0,\ldots,n-1, \tag{2.16}$$

where the parameters $a_{k,i}$, $i=0,1,\ldots,k$, $k=0,\ldots,n-1$ satisfy the following recurrence equations :

[13] Due to the last assumption, the described calibration approach is not highly regarded by numerical analysts (see [51] for a discussion). However, it is well suited in our case studies.

$$\begin{cases} a_{0,0} = 1, \\ a_{k,i} = \frac{\lambda_{k-1}}{\lambda_k - \lambda_i} a_{k-1,i}, \quad i = 0, 1, \ldots, k-1, \; k = 1, \ldots, n-1, \\ a_{k,k} = -\sum_{i=0}^{k-1} a_{k,i} \end{cases} \tag{2.17}$$

Then, we check easily that, the functions $t \mapsto p(t,k)$, $k = 0, \ldots, n-1$ described by (2.16) and (2.17) provide some solutions of the forward PDE. Since it is well-known that these solutions are unique, it means we have obtained explicitly the solutions of the forward PDE, knowing the intensities $(\lambda_k)_{k=0,\ldots,n-1}$. Therefore, using $p(0,k) = 0$ and $\lambda_0 = -\frac{\ln(p(T,0))}{T}$, we can compute iteratively $\lambda_1, \ldots, \lambda_{n-1}$ by solving the univariate non-linear implicit equations $p(T,k) = \sum\limits_{i=0}^{k} a_{k,i} e^{-\lambda_i T}$, or equivalently:

$$\sum_{i=0}^{k-1} a_{k-1,i} e^{-\lambda_i T} \left(\frac{1 - e^{-(\lambda_k - \lambda_i)T}}{\lambda_k - \lambda_i} \right) = \frac{p(T,k)}{\lambda_{k-1}}, \; k = 1, \ldots, n-1. \tag{2.18}$$

It can be seen easily that for any $k \in \{0, \ldots, n-1\}$, $p(T,k)$ is a decreasing function of λ_k, taking value $\lambda_{k-1} \int\limits_0^T p(s,k-1)ds$ for $\lambda_k = 0$ and with the limit equal to zero as λ_k tends to infinity. In other words, the system of equations (2.18) has a unique solution provided that:

$$p(T,k) < \lambda_{k-1} \left(\sum_{i=0}^{k-1} a_{k-1,i} \left(\frac{1 - e^{-\lambda_i T}}{\lambda_i} \right) \right), \; k = 1, \ldots, n-1. \tag{2.19}$$

Note that, in practice, all the intensities λ_k will be different. Thus, starting from the T–default probabilities only, we have found the explicit solutions of the forward equations and the intensities $(\lambda_k)_{k=0,\ldots,n-1}$ that would be consistent with these probabilities.

2.3.3 Other calibrating approaches

Alternative calibrating approaches based on minimization algorithms have been proposed by several authors.

In Herbertsson (2007) [36], the name intensities $\alpha(t, N_t)$ are time homogeneous, piecewise constant in the number of defaults (the node points are given by standard detachment points) and they are fitted to spread quotes by a least square numerical procedure. Arnsdorf and Halperin (2007) [1] propose a piecewise constant parameterization of name intensities (which are referred to as “contagion factors”) in time. When intensities are piecewise linear in the number of defaults too, they use a “multi-dimensional solver” to calibrate onto the observed tranche prices. In the same vein, Frey and Backhaus [30], [31] introduce a parametric form for the function $\lambda(t,k)$, a variant of the "convex counterparty risk model", and fit the parameters to some tranche spreads. Lopatin and Misirpashaev [48] express the loss intensity $\lambda(t,k)$ as a polynomial function of an auxiliary variable involving the number of defaults.

Cont and Minca (2008) [14] propose an alternative method based on the principle of minimum relative entropy. The name intensities has to be chosen in such a way that the loss process is close enough to a simple prior process in the sens of an entropy distance. In the same time, the usual calibration constraints have to be satisfied. However, the main drawback of this approach is the fact that the fitted intensities strongly rely on the choice of the prior.

In the spirit of Dupire [25], Cont, Deguest and Kan [12] show that loss intensities $\lambda\,(t,k)$, $0 \le t \le T$, $k = 0,\dots,n$ can be formulated using prices of put options on the aggregate loss, i.e., $\mathbb{E}[\min(N_t,k)]$, $0 \le t \le T$, $k = 0,\dots,n$. It allows to transform the calibration of the loss intensities into the calibration of the put option values. Given the small number of available quotes, they remark that there can be several sets of put values that are consistent with the market CDO data. Therefore, a calibration algorithm based on quadratic programming is proposed in order to pinpoint a unique set of put values if it exists. They also compare their method with the calibration approaches introduced by Herbertsson [36] and Cont and Minca [14] and show that calibrated intensity surfaces can be significantly different across algorithms.

2.4 Computation of credit deltas through a recombining tree

We now address the computation of CDO tranche deltas with respect to the credit default swap index of the same maturity. As for the hedging instrument, the premium is set at the inception of the deal and remains fixed which corresponds to market conventions. We do not take into account roll dates every six months and trade the same index series up to maturity. Switching from one hedging instrument to another could be dealt with very easily in our framework and closer to market practice but we thought that using the same underlying across the tree would simplify the exposition[14].

2.4.1 Building up a tree

Let us recall that the (fractional) loss at time t is given by $L_t = (1-R)\frac{N_t}{n}$. In what follows, we consider a tranche with attachment point a and detachment point b,

[14] Actually, the credit deltas at inception are the same whatever the choice.

$0 \le a \le b \le 1$. Up to some minor adjustment for the premium leg (see below), the credit default swap index is assimilated to a $[0, 100\%]$ tranche. We denote by $O(N_t)$ the outstanding nominal on a tranche. It is equal to $b-a$ if $L(t) < a$, to $b - L(t)$ if $a \le L(t) < b$ and to 0 if $L(t) \ge b$.

Let us recall that, for a European type payoff the price vector fulfils $V(t, .) = e^{-r(t'-t)}\mathbf{Q}(t,t')V(t',)$ for $0 \le t \le t' \le T$. The transition matrix can be expressed as $\mathbf{Q}(t,t') = \exp(\Lambda(t'-t))$ where Λ is the generator matrix associated with the number of defaults process. Note that, in the time homogeneous framework discussed in the previous section, the generator matrix does not depend on time.

For practical implementation, we will be given a set of node dates $t_0 = 0$, $\ldots, t_i, \ldots, t_{n_s} = T$. For simplicity, we will further consider a constant time step $\Delta = t_1 - t_0 = \cdots = t_i - t_{i-1} = \cdots$; this assumption can easily be relaxed. The most simple discrete time approximation one can think of is $\mathbf{Q}(t_i, t_{i+1}) \simeq Id + \Lambda(t_i) \times (t_{i+1} - t_i)$, which leads to $\mathbb{Q}\left(N_{t_{i+1}} = k+1 \,|N_{t_i} = k\right) \simeq \lambda_k \Delta$ and $\mathbb{Q}\left(N_{t_{i+1}} = k \,|N_{t_i} = k\right) \simeq 1 - \lambda_k \Delta$. For large λ_k, the transition probabilities can become negative. Thus, we will rather use the following approximations :

$$\begin{cases} \mathbb{Q}\left(N_{t_{i+1}} = k+1 \,|N_{t_i} = k\right) \simeq 1 - e^{-\lambda_k \Delta}, \\ \mathbb{Q}\left(N_{t_{i+1}} = k \,|N_{t_i} = k\right) \simeq e^{-\lambda_k \Delta}. \end{cases} \tag{2.20}$$

Given the latter approximations and as illustrated in Figure 2.6, the dynamics of the number of defaults process can be described through a recombining tree.

This idea has also been exploited by van der Voort [63]. One could clearly think of using continuous Markov chain techniques to compute present values of derivative products at hand, but the tree implementation is quite intuitive from a financial point of view as it corresponds to the implied binomial tree of Derman and Kani [23]. Convergence of the discrete time Markov chain to its continuous limit is a rather standard issue and will not be detailed here.

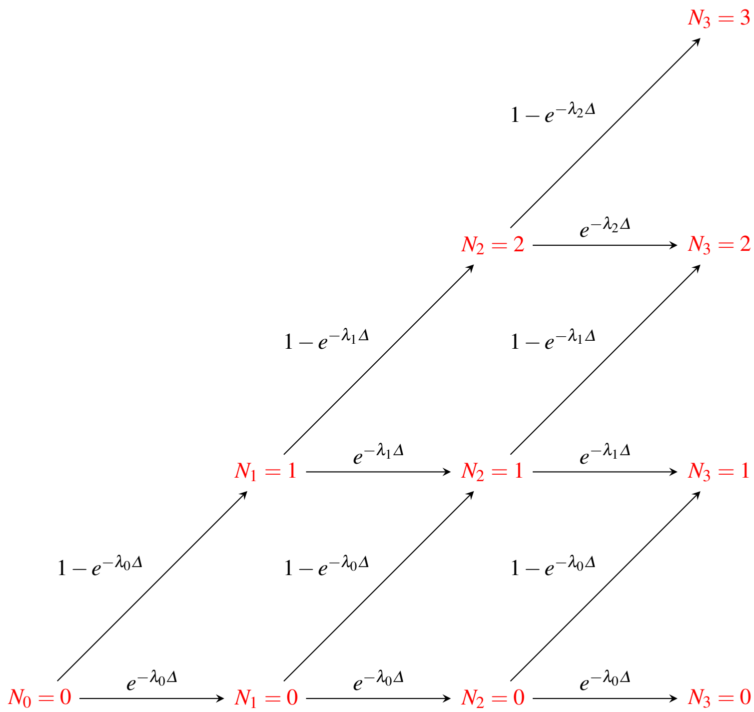

Fig. 2.6 Number of defaults tree

2.4.2 Computation of hedge ratios for CDO tranches

2.4.2.1 Present values of a CDO tranche in the tree nodes

Let us denote by $D(i,k)$ the value at time t_i when $N_{t_i} = k$ of the default payment leg of the CDO tranche[15]. The default payment at time t_{i+1} is equal to $O\left(N_{t_i}\right) - O\left(N_{t_{i+1}}\right)$. Thus, $D(i,k)$ is given by the following recurrence equation[16] :

$$D(i,k) = e^{-r\Delta} \cdot \left(\left(1 - e^{-\lambda_k \Delta}\right) \{D(i+1,k+1) + O(k) - O(k+1)\} \right.$$
$$\left. + \, e^{-\lambda_k \Delta} D(i+1,k) \right). \tag{2.21}$$

Let us now deal with a (unitary) premium leg. We denote the regular premium payment dates by $T_1, \ldots, T_p$ and for simplicity we assume that: $\left\{T_1, \ldots, T_p\right\} \subset$

[15] We consider the value of the default leg immediately after t_i. Thus, we do not consider a possible default payment at t_i in the calculation of $D(i,k)$.

[16] This relation holds for $i = 0, \ldots, n_s - 1$, $k = 0, \ldots, \min(i, n-1)$ and with $D(n_s, k) = 0$ when $k = 0, \ldots, n$ and $D(i,n) = 0$ when $i = n, \ldots, n_s - 1$.

$\{t_0,\ldots,t_{n_s}\}$. Let us consider some date t_{i+1} and set l such that $T_l < t_{i+1} \le T_{l+1}$. Whatever t_{i+1}, there is an accrued premium payment of $\left(O\left(N_{t_i}\right) - O\left(N_{t_{i+1}}\right)\right) \times (t_{i+1} - T_l)$. If $t_{i+1} = T_{l+1}$, i.e., t_{i+1} is a regular premium payment date, there is an extra premium cash-flow at time t_{i+1} of $O\left(N(T_{l+1})\right) \times (T_{l+1} - T_l)$. Thus, if t_{i+1} is a regular premium payment date, the total premium payment is equal to $O\left(N_{t_i}\right) \times (T_{l+1} - T_l)$. Let us denote by $P(i,k)$ the value at time t_i when $N_{t_i} = k$ of the unitary premium leg[17]. If $t_{i+1} \in \left\{T_1,\ldots,T_p\right\}$, $P(i,k)$ is provided by:

$$
\begin{aligned}
P(i,k) = e^{-r\Delta} \cdot \Big(& O(k)\left(T_{l+1} - T_l\right) \\
& + \left(1 - e^{-\lambda_k \Delta}\right) P(i+1,k+1) + e^{-\lambda_k \Delta} P(i+1,k) \Big).
\end{aligned}
\tag{2.22}
$$

If $t_{i+1} \notin \left\{T_1,\ldots,T_p\right\}$, then[18] :

$$
\begin{aligned}
P(i,k) = e^{-r\Delta} \cdot \Big(& \left(1 - e^{-\lambda_k \Delta}\right) \left\{P(i+1,k+1) + \left(O(k) - O(k+1)\right)\left(t_{i+1} - T_l\right)\right\} \\
& + e^{-\lambda_k \Delta} P(i+1,k) \Big).
\end{aligned}
\tag{2.23}
$$

The CDO tranche premium is equal to $\kappa = \frac{D(0,0)}{P(0,0)}$. The value of the CDO tranche (buy protection case) at time t_i when $N_{t_i} = k$ is given by $V(i,k) = D(i,k) - \kappa \cdot P(i,k)$. The equity tranche needs to be dealt with slightly differently since its spread is set to $\kappa = 500$bp. However, the value of the CDO equity tranche is still given by $D(i,k) - \kappa \cdot P(i,k)$.

2.4.2.2 Present values of a CDS index in the tree nodes

As for the credit default swap index, we will denote by $P^I(i,k)$ and $D^I(i,k)$ the values of the premium and default legs. We define the credit default swap index spread at time t_i when $N_{t_i} = k$ by $\kappa^I(i,k) \cdot P^I(i,k) = D^I(i,k)$. The value of the credit default swap index, bought at inception, at node (i,k) is given by $V^I(i,k) = D^I(i,k) - \kappa^I(0,0) \cdot P^I(i,k)$[19]. The default leg of the credit default swap index is computed as a standard default leg of a $(0,100\%)$ CDO tranche. Thus, in the recursion Equation 2.21 giving $D^I(i,k)$, we write the outstanding nominal for k defaults as $O(k) = 1 - \frac{k(1-R)}{n}$, where R is the recovery rate and n the number of names. According to standard market rules, the premium leg of the credit default swap in-

[17] As for the default leg, we consider the value of the premium leg immediately after t_i. Thus, we do not take into account a possible premium payment at t_i in the calculation of $P(i,k)$ either.

[18] Relations 2.22 and 2.23 hold for $i = 0,\ldots,n_s - 1$, $k = 0,\ldots,\min(i,n-1)$ and with $P(n_s,k) = 0$ when $k = 0,\ldots,n$ and $P(i,n) = 0$ when $i = n,\ldots,n_s - 1$.

[19] This is an approximation of the index spread since, according to market rules, the first premium payment is reduced.

dex needs a slight adaptation since the premium payments are based only upon the number of non-defaulted names and do not take into account recovery rates. As a consequence, the outstanding nominal to be used in the recursion equations 2.22 and 2.23 providing $P^I(i,k)$ is such that $O(k) = 1 - \frac{k}{n}$.

2.4.2.3 Computation of credit deltas in the tree nodes

As usual in binomial trees, $\delta(i,k)$ is the ratio of the difference of the option value (at time t_{i+1}) in the upper state ($k+1$ defaults) and lower state (k defaults) and the corresponding difference for the underlying asset. In our case, both the CDO tranche and the credit default swap index are "dividend-baring". For instance, when the number of defaults switches from k to $k+1$, the default leg of the CDO tranche is associated with a default payment of $O(k) - O(k+1)$. Similarly, given the above discussion, when the number of defaults switches from k to $k+1$, the premium leg of the CDO tranche is associated with an accrued premium payment equal to[20]

$$-\kappa 1_{t_{i+1} \notin \{T_1,\ldots,T_p\}} \left(O(k) - O(k+1)\right)(t_{i+1} - T_l). \tag{2.24}$$

Thus, when a default occurs the change in value of the CDO tranche is the outcome of a capital gain of $V(i+1,k+1) - V(i+1,k)$ and of a cash-flow of

$$D(i,k) = \left(O(k) - O(k+1)\right)\left(1 - \kappa 1_{t_{i+1} \notin \{T_1,\ldots,T_p\}} (t_{i+1} - T_l)\right). \tag{2.25}$$

Similarly, when a default occurs the change in value of the credit default swap index is the outcome of a capital gain of $V^I(i+1,k+1) - V^I(i+1,k)$ and a cash-flow of

$$D^I(i,k) = \frac{1-R}{n} - \frac{1}{n}\kappa^I(0,0) 1_{t_{i+1} \notin \{T_1,\ldots,T_p\}} (t_{i+1} - T_l) \tag{2.26}$$

The credit delta of the CDO tranche at node (i,k) with respect to the credit default swap index is thus given by:

$$\delta(i,k) = \frac{V(i+1,k+1) - V(i+1,k) + D(i,k)}{V^I(i+1,k+1) - V^I(i+1,k) + D^I(i,k)}. \tag{2.27}$$

Let us remark that using the previous credit deltas leads to a perfect replication of a CDO tranche within the tree, which is feasible since the approximating discrete market is complete.

In the next section, we compute CDO tranche credit deltas with respect to credit default swap index in two steps. We first calibrate loss intensities from a one factor Gaussian copula loss distribution. It allows us to examine how the correlation between defaults impact credit deltas. We then calibrate loss intensities from a loss

[20] If $t_{i+1} \in \{T_1,\ldots,T_p\}$, the premium payment is the same whether the number of defaults is equal to k or $k+1$. So, it does not appear in the computation of the credit delta.

distribution associated with a market base correlation structure and we compare our "default risk" deltas with some "credit spread" deltas computed on a basis of a bump of credit default swap index spread. We investigate in particular spread deltas computed from the standard market approach and spread deltas recently obtained in [1] and [26].

2.4.3 Model calibrated on a loss distribution associated with a Gaussian copula

In this numerical illustration, the loss intensities λ_k are computed from a loss distribution generated from a one factor homogeneous Gaussian copula model[21]. The correlation parameter is equal to $\rho^2 = 30\%$, the credit spreads are assumed to be all equal to $\kappa = 20$ basis points per annum[22], the recovery rate is such that $R = 40\%$ and the maturity is $T = 5$ years. The number of names is $n = 125$. Figure 2.7 shows the number of defaults distribution.

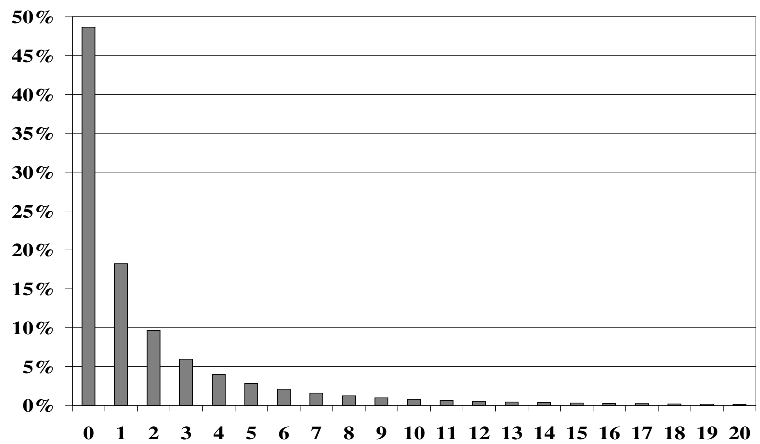

Fig. 2.7 Number of defaults distribution. Number of defaults on the x-axis. $\rho^2 = 30\% : p(5,k)$, $k = 0,\ldots,20$.

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[21] In the homogeneous Gaussian copula model, default times have the same marginal distribution, says F. In that model, default times are defined by $\tau_i = F^{-1}(\Phi(V_i))$, $i = 1,\ldots,n$, where Φ is the standard Gaussian cumulative distribution and $V_1,\ldots,V_n$ are some latent variables such that : $V_i = \rho V + \sqrt{1-\rho^2}\bar{V}_i$, $i = 1,\ldots,n$. The factors $V, \bar{V}_i$, $i = 1,\ldots,n$ are independent standard Gaussian random variables.

[22] Marginal default probabilities have been computed using the classical assumption, under which default times are exponentially distributed with parameter $\frac{\kappa}{1-R}$, i.e., the cumulative distribution of default times at time T is equal to $F(T) = \mathbb{Q}(\tau_1 \leq T) = 1 - \exp\left(-\frac{\kappa}{1-R}T\right)$.

Loss intensities λ_k are calibrated up to $k = 49$ defaults according to the method proposed in Subsection 2.3.2. Under the Gaussian copula assumption, the default probabilities $p(5,k)$ are insignificant[23] for $k > 49$. To avoid numerical difficulties, we computed the remaining λ_k ($k > 49$) by linear extrapolation[24].

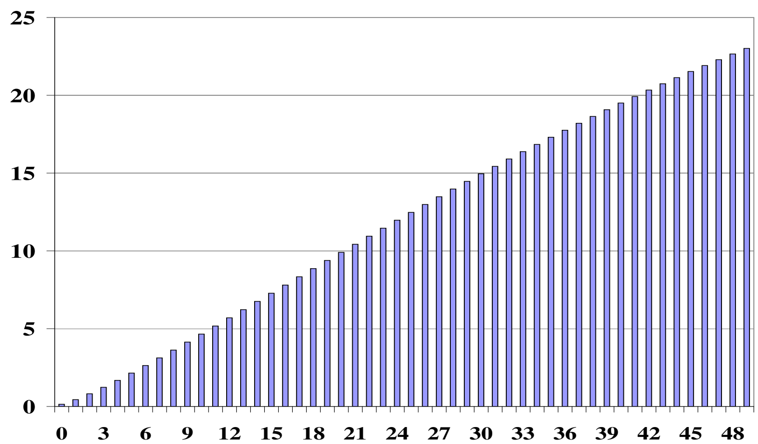

Fig. 2.8 Loss intensities λ_k, $k = 0,\ldots,49$.

As can be seen from Figure 2.8, loss intensities change almost linearly with respect to the number of defaults. Let us also remark that such rather linear behaviour of loss intensities can be found in [48]. Our results can also be related to the analysis of Ding et al. [24] who deal with a dynamic model where the loss intensity is actually linear in the number of defaults.

Table 2.1 shows the dynamics of the credit default swap index spreads $\kappa^I(i,k)$ along the nodes of the tree. The continuously compounded default free rate is $r = 3\%$ and the time step is $\Delta = \frac{1}{365}$. It can be seen that default arrivals are associated with rather large jumps of credit spreads. For instance, if a (first) default occurs after a quarter, the credit default swap index spread jumps from 18 bps to 70 bps. An extra default by this time leads to an index spread of 148 bps.

The credit deltas with respect to the credit default swap index $\delta(i,k)$ have been computed for the $(0-3\%)$ and the $(3-6\%)$ CDO tranches (see Table 2.2 and Table 2.3). As for the equity tranche, it can be seen that the credit deltas are positive and-decrease up to zero. This is not surprising given that a buy protection equity tranche

[23] $\sum_{k\geq 50} p(5,k) \simeq 3\times 10^{-4}$, $p(5,50) \simeq 3.2\times 10^{-5}$ et $p(5,125) \simeq 4\times 10^{-12}$.

[24] We checked that various choices of loss intensities for high number of defaults had no effect on the computation of deltas. Let us stress that this applies for the Gaussian copula case since the loss distribution has thin tails. For the market case example, we proceeded differently.

Nb Defaults	**Weeks**			
	0	**14**	**56**	**84**
0	20	18	14	13
1	0	70	54	46
2	0	148	112	93
3	0	243	182	150
4	0	350	261	215
5	0	466	347	285
6	0	589	437	359
7	0	719	531	436
8	0	856	630	516
9	0	997	732	598
10	0	1142	839	683

Table 2.1 Dynamics of credit default swap index spread $\kappa^I(i,k)$ in basis points per annum.

involves a short put position over the aggregate loss with a 3% strike. This is associated with positive deltas, negative gammas and thus decreasing deltas. When the number of defaults is above 6, the equity tranche is exhausted and the deltas obviously are equal to zero.

Nb Defaults	**Outstanding Nominal**	**Weeks**			
		0	**14**	**56**	**84**
0	**3.00%**	0.538	0.591	0.755	0.859
1	**2.52%**	0	0.238	0.381	0.508
2	**2.04%**	0	0.074	0.137	0.212
3	**1.56%**	0	0.026	0.044	0.070
4	**1.08%**	0	0.011	0.017	0.024
5	**0.60%**	0	0.005	0.007	0.009
6	**0.12%**	0	0.001	0.001	0.001
7	**0.00%**	0	0	0	0

Table 2.2 Delta of the $[0-3\%]$ equity tranche with respect to the credit default swap index.

At inception, the credit delta of the equity tranche is equal to 54% whilst it is only equal to 25% for the $[3-6\%]$ tranche which is deeper out of the money (see Table 2.3). Moreover, the $[3-6\%]$ CDO tranche involves a call spread position over the aggregate loss. As a consequence the credit deltas are positive and firstly increase (positive gamma effect) and then decrease (negative gamma) up to zero as soon as the tranche is fully amortized.

Given the recovery rate assumption of 40%, the outstanding nominal of the $[3-6\%]$ is equal to 3% for six defaults and to 2.64% for seven defaults. One might thus think that at the sixth default the $[3-6\%]$ tranche should behave almost like an equity tranche. However, as can be seen from Table 2.3, the credit delta of the $[3-6\%]$ tranche is much lower: around 1% instead of 60%. This is due to dramatic shifts in credit spreads when moving from the no-defaults to the six defaults state

Nb Defaults	**Outstanding Nominal**	**Weeks**			
		0	**14**	**56**	**84**
0	**3.00%**	0.255	0.254	0.219	0.171
1	**3.00%**	0	0.280	0.349	0.357
2	**3.00%**	0	0.167	0.294	0.389
3	**3.00%**	0	0.068	0.158	0.265
4	**3.00%**	0	0.026	0.065	0.128
5	**3.00%**	0	0.014	0.027	0.053
6	**3.00%**	0	0.010	0.016	0.025
7	**2.64%**	0	0.008	0.011	0.015
8	**2.16%**	0	0.006	0.008	0.010
9	**1.68%**	0	0.004	0.005	0.007
10	**1.20%**	0	0.003	0.003	0.004
11	**0.72%**	0	0.002	0.002	0.002
12	**0.24%**	0	0.001	0.001	0.001
12	**0.00%**	0	0	0	0

Table 2.3 Deltas of the $[3-6\%]$ with respect to the credit default swap index.

(see Table 2.1). In the latter case, the expected loss on the tranche is much larger, which is consistent with smaller deltas given the call spread payoff.

2.4.4 Dependence of hedging strategies upon the correlation parameter

Let us recall that the recombining tree is calibrated on a loss distribution over a given time horizon. The shape of the loss distribution depends critically upon the correlation parameter which was set up to now to $\rho^2 = 30\%$. Decreasing the dependence between default events leads to a thinner right-tail of the loss distribution and smaller contagion effects. We detail here the effects of varying the correlation parameter on the hedging strategies. For simplicity, we firstly focus the analysis on the equity tranche and shift the correlation parameter from 30% to 10%. It can be seen from Tables 2.2 and 2.4 that the credit deltas are much higher in the latter case. After 14 weeks, prior to the first default, the credit delta is equal to 59% for a 30% correlation and to 96% when the correlation parameter is equal to 10%[25].

To further investigate how changes in correlation levels alter credit deltas, we computed the market value of the default leg of the equity tranche at a 14 weeks horizon as a function of the number of defaults under different correlation assumptions (see Figure 2.9). The market value of the default leg, on the y - axis, is computed as the sum of expected discounted cash-flows posterior to this 14 weeks horizon date and the accumulated defaults cash-flows paid before[26]. We also plotted the accu-

[25] Let us remark that credit deltas can be above one in the no default case. This is due to the amortization scheme of the premium leg. We detail in the next section the impact of the premium leg on credit deltas.

[26] For simplicity, we neglected the compounding effects over this short period.

Nb Defaults	Outstanding Nominal	Weeks			
		0	14	56	84
0	**3.00%**	0.931	0.960	1.009	1.058
1	**2.52%**	0	0.694	0.785	0.910
2	**2.04%**	0	0.394	0.485	0.645
3	**1.56%**	0	0.179	0.233	0.352
4	**1.08%**	0	0.072	0.092	0.145
5	**0.60%**	0	0.027	0.032	0.046
6	**0.12%**	0	0.004	0.005	0.007
7	**0.00%**	0	0	0	0

Table 2.4 Deltas of the $[0-3\%]$ *equity* tranche with respect to the credit default swap index, $\rho^2 = 10\%$.

mulated losses which represent the intrinsic value of the equity tranche default leg. Unsurprisingly, we recognize some typical concave patterns associated with a short put option payoff.

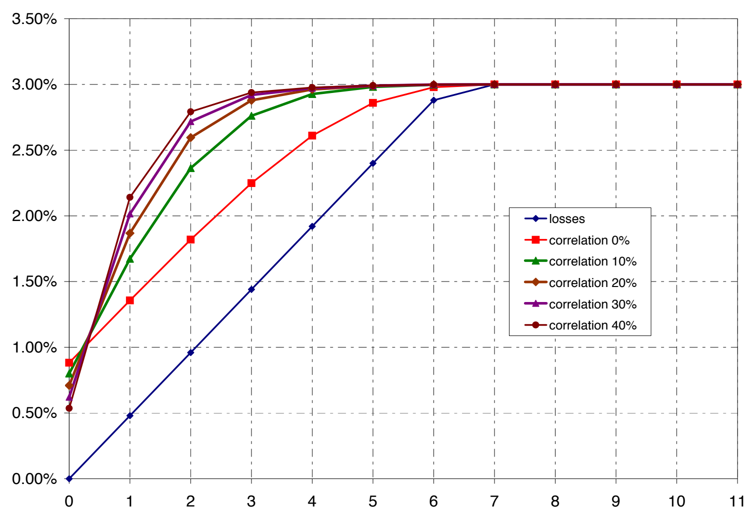

Fig. 2.9 Market value of equity default leg under different correlation assumptions. Number of defaults on the x - axis.

As can be seen from Figure 2.9, prior to the first default, the value of the default leg of the equity tranche decreases as the correlation parameter increases from 0% to 40%. However, after the first default the ordering of default leg values is reversed. This can be easily understood since larger correlations are associated with larger jumps in credit spreads at default arrivals and thus larger changes in the expected discounted cash-flows associated with the default leg of the equity tranche[27].

Therefore, varying the correlation parameter is associated with two opposite mechanisms :

- The first one is related to a typical negative vanna effect[28]. Increasing correlation lowers loss "volatility" and leads to smaller expected losses on the equity tranche. In a standard option pricing framework, this should lead to an increase in the credit delta of the short put position on the loss.
- This is superseded by the shifts due to contagion effects. Increasing correlation is associated with bigger contagion effects and thus larger jumps in credit spreads at the arrival of defaults. This, in turn leads to a larger jump in the market value of the credit index default swap. Let us recall that the default leg of the equity tranche exhibits a concave payoff and thus a negative gamma. As a consequence the credit delta, i.e. the ratio between the change in value of the option and the change in value of the underlying, decreases.

2.4.5 *Model calibrated on a loss distribution associated with CDO tranche quotes*

Up to now, the probabilities of number of defaults were computed thanks to a Gaussian copula and a single correlation parameter. In this example, we use a steep upward sloping *base correlation curve* for the iTraxx, typical of June 2007, as an input to derive the distribution of the probabilities of number of defaults (see Table 2.5). The maturity is still equal to 5 years, the recovery rate to 40% and the credit spreads to 20 bps. The default-free rate is now equal to 4%.

Base Tranches	**[0-3%]**	**[0-6%]**	**[0-9%]**	**[0-12%]**	**[0-22%]**
Base Correlations	18%	28%	36%	42%	58%

Table 2.5 Base correlations with respect to attachment points (iTraxx Juin 2007).

Rather than spline interpolation of base correlations, we used a parametric model of the 5 year loss distribution to fit the market quotes and compute the probabilities of the number of defaults. This produces arbitrage free and smooth distributions

[27] Let us remark that the larger the correlation the larger the change in market value of the default leg of the equity tranche at the arrival of the first default. Indeed, in a high correlation framework, this default means relatively higher default likelihood for the surviving names. This is not inconsistent with the previous results showing a decrease in credit deltas when the correlation parameter increases. The credit delta is the ratio of the change in value in the equity tranche and of the change in value in the credit default swap index. For a larger correlation parameter, the change in value in the credit default swap index is also larger due to magnified contagion effects.

[28] We recall that in option pricing, the vanna is the sensitivity of the delta to a unit change in volatility.

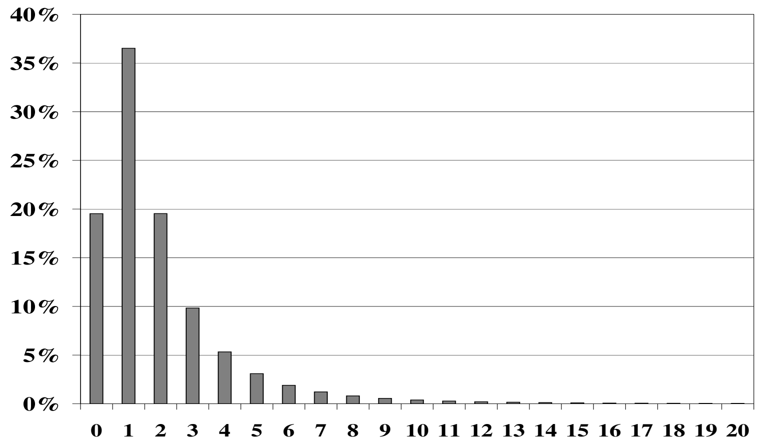

Fig. 2.10 Number of defaults distribution obtained from the *base correlation* structure described in Table 2.5. Number of defaults on the x - axis.

that ease the calculation of the loss intensities[29]. Figure 2.10 shows the number of defaults distribution. This is rather different from the 30% flat correlation Gaussian copula case both for small and large losses. For instance, the probability of no defaults dropped from 48.7% to 19.5% while the probability of a single default rose from 18.2% to 36.5%. Let us stress that these figures are for illustrative purpose. The market does not provide direct information on first losses and thus the shape of the left tail of the loss distribution is a controversial issue. As for the right-tail, we have $\sum_{k\geq 50} p(5,k) \simeq 1.4\times 10^{-3}$ and $p(5,50) \simeq 3.3\times 10^{-6}$, $p(5,125) \simeq 1.38\times 10^{-3}$. The cumulative probabilities of large number of defaults are larger, compared with the Gaussian copula case. The probability of the names defaulting altogether is also quite large, corresponding to some kind of Armageddon risk. Once again these figures need to be considered with caution, corresponding to high senior and super-senior tranche premiums and disputable assumptions about the probability of all names defaulting.

Figure 2.11 shows the loss intensities calibrated onto market inputs compared with the loss intensities based on Gaussian copula inputs up to 39 defaults . As can be seen, the loss intensity increases much quicker with the number of defaults as compared with the Gaussian copula approach. The average relative change in the loss intensities is equal to 19% when it is equal to 16% when computed under the Gaussian copula assumption. Unsurprisingly, a steep base correlation curve is associated with fatter upper tails of the loss distribution and magnified contagion effects.

[29] We also computed the number of defaults distribution using entropic calibration. Although we could still compute loss intensities, the pattern with respect to the number of defaults was not monotonic. Depending on market inputs, direct calibration onto CDO tranche quotes can lead to shaky figures.

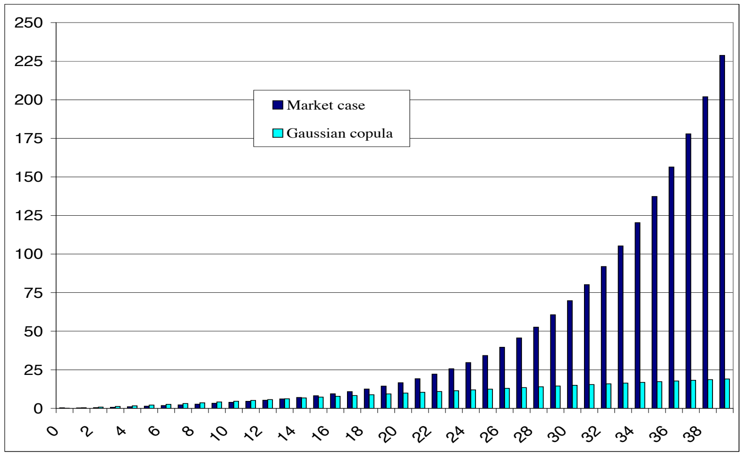

Fig. 2.11 Loss intensities for the Gaussian copula and market case examples. Number of defaults on the x - axis.

Table 7 shows the dynamics of the credit default swap index spreads $\kappa^I(i,k)$ along the nodes of the tree. As for tree implementation, the time step is still $\Delta = \frac{1}{365}$. Let us remark that up to 12 defaults, loss intensities calibrated from market inputs are on the whole smaller than in the Gaussian copula case. Then, the contagion effect is smaller than in the flat 30% correlation Gaussian copula in low default states and greater for high default states. Unsurprisingly, market quotes lead to smaller index spreads up to 2 defaults at 14 weeks (see Tables 2.1 and 2.6). This is also coherent with Figure 2.12 where the conditional expected losses in the two approaches cross each other at the third default. However, as mentioned above, this detailed pattern has to be considered with caution, since it involves the probabilities of 0, 1 and 2 defaults which are not directly observed in the market. After 2 defaults, credit spreads become definitely larger when calibrated from market inputs.

Nb Defaults	**Weeks**			
	0	**14**	**56**	**84**
0	20	19	17	16
1	0	31	23	20
2	0	95	57	43
3	0	269	150	98
4	0	592	361	228
5	0	1022	723	490
6	0	1466	1193	905
7	0	1870	1680	1420
8	0	2243	2126	2423
9	0	2623	2534	2423
10	0	3035	2939	2859

Table 2.6 Dynamics of credit default swap index spread $\kappa^I(i,k)$ in basis points per annum.

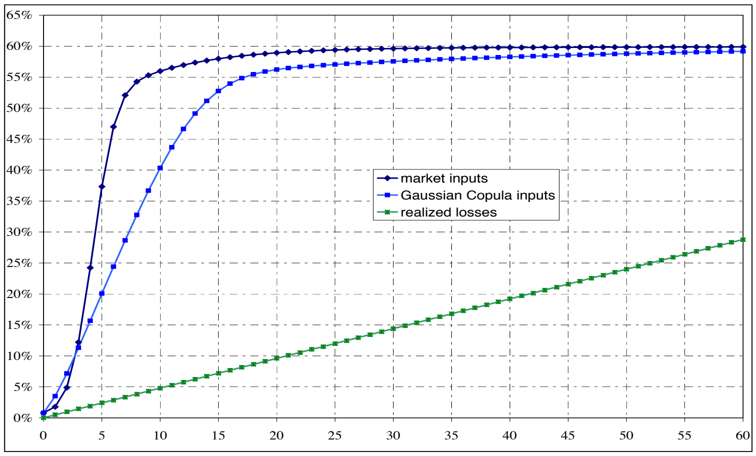

Fig. 2.12 Expected losses on the credit portfolio after 14 weeks over a five year horizon (y - axis) with respect to the number of defaults (x - axis) using market and Gaussian copula inputs.

Thanks to Figure 2.12 we can investigate the credit spread dynamics when using market inputs. We plotted the conditional (with respect to the number of defaults) expected loss $E\left[L_T \left|N_t\right.\right]$ for $T = 5$ years and $t = 14$ weeks for the previous market inputs and for the 30% flat correlation Gaussian copula case. The conditional expected loss is expressed as a percentage of the nominal of the portfolio[30]. We also plotted the accumulated losses on the portfolio. The expected losses are greater than the accumulated losses due to positive contagion effects. There are some dramatic differences between the Gaussian copula and the market inputs examples. In the Gaussian copula case, the expected loss is almost linear with respect to the number of defaults in a wide range (say up to 15 defaults). The pattern is quite different when using market inputs with huge non linear effects. This shows large contagion effects after a few defaults as can also be seen from Table 2.6 and Figure 2.11. This rather explosive behaviour was also observed by Herbertsson [35], Tables 3 and 4 and by Cont and Minca [14], Figures 1 and 3. In Lopatin and Misirpashaev [48], the contagion effects are also magnified when using market data, compared with Gaussian copula inputs.

Table 2.7 shows the dynamic deltas associated with the equity tranche. We notice that the credit deltas drop quite quickly to zero with the occurrence of defaults. This is not surprising given the surge in credit spreads and dependencies after the first default (see Figure 2.12): after only a few defaults the equity tranche is virtually exhausted.

It is noteworthy that the credit deltas $\delta(i,k)$ can be decomposed into a default leg delta $\delta_d(i,k)$ and a premium leg delta $\delta_p(i,k)$ as follows :

$$\delta(i,k) = \delta_d(i,k) - \kappa\delta_p(i,k), \tag{2.28}$$

[30] Thus, given a recovery rate of 40%, the maximum expected loss is equal to 60%.

Nb Defaults	**Outstanding Nominal**	**Weeks**			
		0	**14**	**56**	**84**
0	**3.00%**	0.645	0.731	0.953	1.038
1	**2.52%**	0	0.329	0.584	0.777
2	**2.04%**	0	0.091	0.197	0.351
3	**1.56%**	0	0.023	0.045	0.090
4	**1.08%**	0	0.008	0.011	0.018
5	**0.60%**	0	0.004	0.003	0.004
6	**0.12%**	0	0.001	0.001	0.001
7	**0.00%**	0	0	0	0

Table 2.7 Delta of the $[0-3\%]$ equity tranche with respect to the credit default swap index.

where :

$$\delta_d(i,k) = \frac{D(i+1,k+1) - D(i+1,k) + O(k) - O(k+1)}{V^I(i+1,k+1) - V^I(i+1,k) + D^I(i,k)} \tag{2.29}$$

and

$$\delta_p(i,k) = \frac{P(i+1,k+1) - P(i+1,k) + (O(k) - O(k+1))\, 1_{t_{i+1} \notin \{T_1,\dots,T_p\}}\,(t_{i+1} - T_l)}{V^I(i+1,k+1) - V^I(i+1,k) + D^I(i,k)}. \tag{2.30}$$

Tables 2.8 and 2.9 detail the credit deltas associated with the default and premium legs of the equity tranche. As can be seen from Table 2.7, credit deltas for the equity tranche may be slightly above one when no default has occurred. Table 2.9 shows that this is due to the amortization scheme of the premium leg which is associated with significant negative deltas. Let us recall that premium payments are based on the outstanding nominal. Arrival of defaults thus reduces the commitment to pay. Furthermore, the increase in credit spreads due to contagion effects involves a decrease in the expected outstanding nominal. When considering the default leg only, we are led to credit deltas that actually remain within the standard 0%-100% range. The default leg delta of the equity tranche with respect to the credit default swap index is initially equal to 54.1%. Let us also remark that credit deltas of the default leg gradually increase with time which is consistent with a decrease in time value.

2.4.6 Comparison with standard market practice

We further examine the credit deltas of the different tranches at inception. These are compared with the deltas as computed by market participants under the previous base correlation structure assumption (see Table 2.10). These market deltas are calculated by bumping the credit curves by 1 basis point and computing the changes in present value of the tranches and of the credit default swap index. Once the credit curves are bumped, the moneyness varies, but the market practice is to keep constant

Nb Defaults	**Outstanding Nominal**	**Weeks**			
		0	**14**	**56**	**84**
0	**3.00%**	0.541	0.617	0.823	0.910
1	**2.52%**	0	0.279	0.510	0.690
2	**2.04%**	0	0.072	0.166	0.304
3	**1.56%**	0	0.016	0.034	0.072
4	**1.08%**	0	0.004	0.006	0.012
5	**0.60%**	0	0.002	0.002	0.002
6	**0.12%**	0	0.001	0.000	0.000
7	**0.00%**	0	0	0	0

Table 2.8 Delta of the default leg of the $[0-3\%]$ equity tranche with respect to the credit default swap index ($\delta_d(i,k)$).

Nb Defaults	**Outstanding Nominal**	**Weeks**			
		0	**14**	**56**	**84**
0	**3.00%**	-0.104	-0.113	-0.130	-0.128
1	**2.52%**	0	-0.050	-0.074	-0.087
2	**2.04%**	0	-0.018	-0.031	-0.047
3	**1.56%**	0	-0.007	-0.011	-0.018
4	**1.08%**	0	-0.004	-0.004	-0.006
5	**0.60%**	0	-0.002	-0.002	-0.002
6	**0.12%**	0	-0.001	0.000	0.000
7	**0.00%**	0	0	0	0

Table 2.9 Deltas of the premium leg of the $[0-3\%]$ equity tranche with respect to the credit default swap index ($\kappa\delta_p(i,k)$).

the base correlations when recalculating the CDO tranches. This corresponds to the so-called “sticky strike” rule. The delta is the ratio of the change in present value of the tranche to the change in present value of the credit default swap index divided by the tranche’s nominal. For example, a credit delta of an equity tranche previously equal to one would now lead to a figure of 33.33.

Tranches	**[0-3%]**	**[3-6%]**	**[3-9%]**	**[9-12%]**	**[12-22%]**
Market deltas	27	4.5	1.25	0.6	0.25
Model deltas	21.5	4.63	1.63	0.9	0.6

Table 2.10 Market delta spreads and model deltas (a default event) at inception.

First of all we can see that the outlines are roughly the same, which is already noticeable since the two approaches are completely different. Then, we can remark that the model deltas are smaller for the equity tranche as compared with the market deltas, while there are larger for the other tranches.

These discrepancies can be understood from the dynamics of the dependence between defaults embedded in the Markovian contagion model. Figure 2.13 shows the base correlation curves at a 14 weeks horizon, when the number of defaults is equal to zero, one or two. We can see that the arrival of the first defaults is associated with parallel shifts in the base correlation curves. This increase in dependence counterbalances the increase of credit spreads and expected losses on the equity tranche and lowers the credit delta. The model deltas can be thought of as the "sticky implied tree" model deltas of Derman [22]. These are suitable in a regime of fear corresponding to systematic credit shifts.

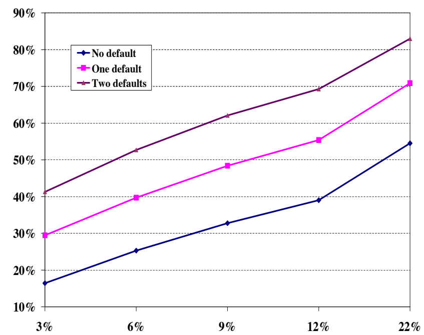

Fig. 2.13 Dynamics of the base correlation curve with respect to the number of defaults. Detachment points on the x - axis. Base correlations on the y - axis.

The summer 2007 credit crisis provides some evidence that implied correlations tend to increase with credit spreads and thus with expected losses. Figure 2.14 shows the dynamics of the five year iTraxx credit spread and of the implied correlation of the equity tranche. Over this period the correlation between the two series was equal to 91%. This clearly favours the contagion model and once again suggests a flaw in the "sticky strike" market practice.

2.4.7 Comparison with deltas computed in other dynamic credit risk models

We also thought that it was insightful to compare our model deltas and the results provided by Arnsdorf and Halperin (2007) [1], Figure 7 (see Table 2.11).

Tranches	**[0-3%]**	**[3-6%]**	**[3-9%]**	**[9-12%]**	**[12-22%]**
Maket deltas	26.5	4.5	1.25	0.65	0.25
BSLP model deltas	21.9	4.81	1.64	0.79	0.38

Table 2.11 Market and model deltas as in Arnsdorf and Halperin [1].

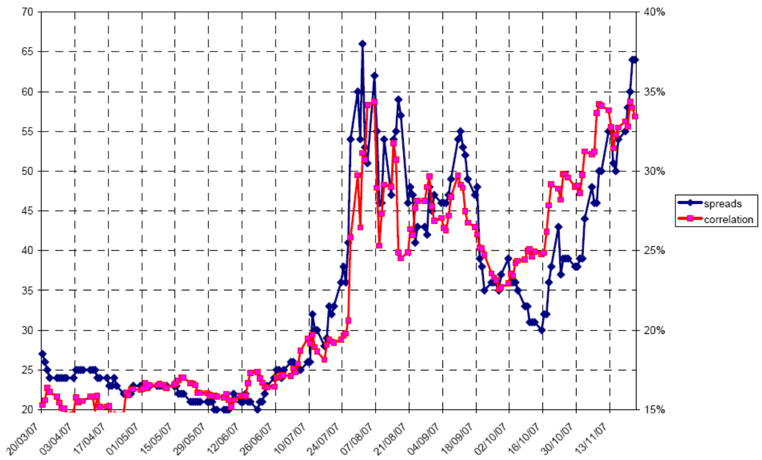

Fig. 2.14 Credit spreads on the five years iTraxx index (Series 7) in bps on the left axis. Implied correlation on the equity tranche on the right axis.

The market conditions are slightly different since the computations were done in March 2007, thus the maturity is slightly smaller than five years. The market deltas are quoted deltas provided by major trading firms. We can see that these are quite close to the previous market deltas since the computation methodology involving Gaussian copula and base correlation is quite standard. The BSLP[31] model deltas (corresponding to "model B" in [1]) have a different meaning from ours: there are related to credit spread deltas rather that then default risk deltas and are not related to a dynamic replicating strategy. However, it is noteworthy that the model deltas in [1] are quite similar to ours, and thus rather far away from market deltas. Though this is not a formal proof, it appears from Figure 2.9, that (systemic) gammas are rather small prior to the first default. If we could view a shock on the credit spreads as a small shock on the expected loss while a default event induces a larger shock (but not so large given the risk diversification at the index level) on the expected loss, the similarity between the different model deltas are not so surprising. As above, model deltas are lower for the equity tranche and larger for the other tranches, when compared with market deltas.

We also compare our model deltas with credit deltas obtained by Eckner (2007) [26], Table 5. Eckner (2007) model relies on an affine specification of default intensities (AJD model). Conditionally on the path of default intensities, default times are independent (i.e. there are no contagion effects at default times). The model is parametric with respect to the term structure of credit spreads and to CDO tranches. Calibration of the model parameters to credit spreads and liquid tranche quotes on

[31] *Bivariate Spread-Loss Portfolio model*.

the CDX NA IG5 index in December 2005 is provided and hedge ratios with respect to the credit default swap index are then computed. The sensitivities of CDO tranche and index prices are computed with respect to a uniform and relative shift of individual intensities. The approach can be extended in order to compute different hedge ratios with respect to the single name default swaps. However, the overall procedure, including the calibration and the computation of individual hedge ratios is likely to be rather involved.

In Table 2.12, the deltas obtained in the AJD intensity model can be compared with those computed from the Gaussian copula model and those computed within a contagion model calibrated to the same data set.

Tranches	**[0-3%]**	**[3-7%]**	**[7-10%]**	**[10-15%]**	**[15-30%]**
Market deltas	18.5	5.5	1.5	0.8	0.4
AJD model deltas	21.7	6.0	1.1	0.4	0.1
Contagion model deltas	17.9	6.3	2.5	1.3	0.8

Table 2.12 Market deltas, "intensity" model credit deltas in Eckner (2007) and contagion model deltas.

Even though the approaches are completely different, once again the outlines are quite similar. Moreover, we can remark that the equity tranche deltas computed by Eckner are higher compared with the market deltas.

Another empirical comparison of various hedging strategies has recently been proposed by Cont and Kan (2008) [13]. This study provides several interesting observations related to the hedging of index CDO tranches, extending the ones presented in this chapter.

Chapter 3
Conclusion

Areski Cousin, Monique Jeanblanc, Jean-Paul Laurent

In this chapter, we were able to show that a CDO tranche payoff can be perfectly replicated with a self-financed strategy based on the underlying credit default swaps. This extends to any payoff which depends only upon defaults arrivals, such as basket default swaps, but does not address the issue of tranche options for instance. Clearly, the previous replication result is model dependent and relies on two critical assumptions. First, we preclude the possibility of simultaneous defaults. In other words, default times can be ordered from the first to the last default time. Hedging against simultaneous defaults would require trading credit default swaps contingent on several defaults, which are not currently traded in the market. The other important assumption, which is likely to be more questionable, is that credit default swap premiums are adapted to the filtration of default times, denoted $\mathbb{H}$, which therefore can be seen as the relevant information set on economic grounds. As a consequence, default swap premiums are deterministic between two default times. Our framework corresponds to a pure contagion model, where the arrival of defaults leads to jumps in the credit spreads of survived names, the magnitude of which depending upon the considered names and the whole history of defaults up to the current time. These jumps can be related to the derivatives of the joint survival function of default times. The dynamics of replicating prices of CDO tranches follows the same way. In other words, we only deal with default risks and not with spread risks. At a given point in time, there are only p sources of risk, related to the default occurrence of the p non defaulted names and we can trade the corresponding p credit default swaps. This provides the intuition of the completeness of the market following the rule of the thumb, "as many hedging instruments as sources of risk". The hedging strategy deals thus with default risks only and not with credit spread risks. Even though the underlying assumptions may look too restrictive, given the risk management and regulatory issues related to CDOs, we think that it may prove useful to rely on benchmark models where the hedging can be fully described and analyzed in a dynamical way.

Unsurprisingly, the possibility of perfect hedging is associated with a martingale representation theorem under the filtration of default times. Subsequently, we exhibit a new probability measure under which the short term credit spreads (up to some scaling factor due to positive recovery rates) are the intensities associated with the corresponding default times. For the ease of presentation, we introduced first some instantaneous default swaps as a convenient basis of hedging instruments. Eventually, we can exhibit a replicating strategy of a CDO tranche payoff with respect to actually traded credit default swaps, for instance, with the same maturity as the CDO tranche. Let us note that no Markovian assumption is required for the existence of such a replicating strategy. Therefore the aggregate loss may not be a Markov process either. Since we dealt first with the dynamics of individual defaults, we are typically in a bottom-up model and no homogeneity assumption, such as equal credit spreads across names is required.

However, when going to implementing actual hedging strategies, one needs extra assumptions, both for the implementation to be feasible and to cope with quoted CDO tranches. We therefore consider the simplest way to specialize the above model: we assume that all pre-default intensities are equal and only depend on the current number of defaults. We also assume that all recovery rates are constant across names and time. In that framework, it can be shown that the aggregate loss process is a homogeneous Markov chain, more precisely a pure death process (thanks to the no simultaneous defaults assumption). The intensity associated with the Markov chain is simply the pre-default intensity times the number of non-defaulted names. Thanks to these restrictions, the model involves as many unknown parameters as the number of underlying names. On the other hand, the knowledge of upfront premiums of equity CDO tranches with different maturities and detachment points (and given some recovery rate) is equivalent to the knowledge of marginal distributions of the number of defaults at different time horizons. Thanks to the forward Kolmogorov equations, one can then perfectly compute the intensities of the aggregate loss process or the pre-default intensities of the names. Such fully calibrated and Markov model is also known as the local intensity model, the simplest form of aggregate loss models. As in local volatility models in the equity derivatives world, there is a perfect match of unknown parameters from a complete set of CDO tranches quotes. In other words, the model is fully specified from market inputs, which is clearly a desirable property, since given some market inputs, we deal with a single model and not with a family of parameterized models. The numerical implementation can be achieved through a binomial tree, well-known to finance people or by means of Markov chain techniques. We provide some examples and show that the market quotes of CDOs are associated with pronounced contagion effects. We can therefore explain the dynamics of the amount of hedging CDS and relate them to deltas computed by market practitioners. The figures are hopefully roughly the same, the discrepancies being mainly explained by contagion effects leading to an increase of dependence between default times after some defaults.

However, one cannot unfortunately observe a complete set of CDO tranche premiums. The set of local intensities consistent with the actually CDO tranches quotes is not a singleton. For a complete specification, one needs to introduce some extra assumptions: either, one can constrain the shape of intensities, for instance assume that there are piecewise constant with respect to the number of defaults with shifts associated to detachment points of traded tranches. Otherwise, as an intermediary step, we may think of fitting some marginal distributions of aggregate losses to CDO tranche quotes or use interpolation techniques consistent with the increase and concavity of the expected loss on equity tranches. Numerical examples in this chapter are constructed under the second approach. Unfortunately, for practical purpose, the computed deltas and thus hedging performance seem rather sensitive to the calibration technique.

One may compare the proposed framework with the standard structural approach, where default time of a given name is the first hitting time of a barrier by a Brownian motion associated with the asset process of the corresponding name. In that structural approach, dependence between default times stems from the correlation between the Brownian motions. In the latter framework, quite similar to a multivariate Black-Scholes setting, CDS are barrier-options and it is also possible to replicate a CDO tranche payoff by dynamically trading the CDS. While the former Markov chain approach focused on default risk, neglecting credit spread risk, the structural approach only deals with credit spread risk. Defaults are predictable and do not constitute an extra source of risk. On the other hand, a structural model can be well approximated in most cases by a one period structural model, where crossing the default barrier is only considered at maturity. This is known to be equivalent to the Gaussian copula model commonly used by practitioners. As mentioned above, an interesting feature is that the deltas with respect to underlying credit default swaps have the same order of magnitude in the two approaches.

However, extending the scope of the approach would result in adding extra complexity, both on mathematical grounds and regarding the specification of credit spreads dynamics. For instance, if we were to introduce some Brownian risks on top of jump to default risks, it is not clear how defaults would drive the volatility of credit spreads. The uncertainty with respect to this substantial model risk is likely to offset the benefit of dealing with credit spread and default risk altogether. At the time being, extra-complexity conveys the risk of darkening the risk management picture and providing a false sense of security. A better understanding of the multivariate dynamics of defaults and credit spreads is required before going any further. Another, more down to earth issue, but of practical importance is related to the set of hedging instruments. Given n names, one can think of using two credit default swaps of different maturities for each underlying name to cope both with default and credit spread risks. This induces extra complexity in implementing hedging strategies.

A more easy to reach extension of the previous framework consists in relaxing the homogeneity of names assumption, while remaining in a pure default setting. For instance, one could think of two homogeneous groups of names, say belonging to two different geographical regions, the intensities depending both upon the number of survived names in each group. This results in a two dimensional Markov chain, since the portfolio state is characterized by the number of survived names in each group. We should then be able to discriminate CDS deltas for names within each subgroup. Let us note that given that we rely upon a bottom-up approach, once calibrated onto liquid CDO tranche quotes, one would be able to consistently price CDO tranches on any sub-portfolio, thus solving the difficult issue of bespoke tranche pricing.

Another possible and easy to implement extension of our setting consists in using a recovery rate depending upon the number of defaults. The easiest way to proceed is to assume some linear (and most likely negative) dependence with respect to the number of defaults in the portfolio. Such assumption will tend to raise the probability of large losses and ease the calibration to the senior tranches.

Eventually, we would like to stress that the approach described in this chapter should be fruitful in computing so called values on defaults. These assess the magnitude of losses on a portfolio (possibly including CDO tranches) after a default occurs. Usually, market practitioners do not take into account credit contagion effects associated with shifts of credit spreads of survived names, which can lead to gross misestimation of credit risk reserves. This can be easily dealt with in our framework.

References

1. Arnsdorf, M., Halperin, I.: BSLP: Markovian bivariate spread-loss model for portfolio credit derivatives (2007). working paper, JP Morgan
2. Arvanitis, A., Laurent, J.P.: On the edge of completeness. RISK **October**, 61–65 (1999)
3. Assaf, D., Langberg, N.A., Savits, T.H., Shaked, M.: Multivariate phase-type distributions. Operations Research **32**(3), 688–702 (1984)
4. Azizpour, S., Giesecke, K.: Self-exciting corporate defaults: contagion vs. frailty (2008). working paper, Stanford University
5. Beumee, J., Brigo, D., Schiemert, D., Stoyle, G.: Charting a course through the CDS Big Bang (2009). Quantitative research, FitchSolutions
6. Bielecki, T., Jakubowski, J., Vidozzi, A., Vidozzi, L.: Study of dependence for some stochastic processes (2007). Working paper
7. Bielecki, T., Jeanblanc, M., Rutkowski, M.: Hedging of basket credit derivatives in default swap market. Journal of Credit Risk **3**(1), 91–132 (2007)
8. Bielecki, T.R., Rutkowski, M.: Credit Risk: Modeling, Valuation and Hedging. Springer (2004)
9. Blanchet-Scalliet, C., Jeanblanc, M.: Hazard rate for credit risk and hedging defaultable contingent claims. Finance and Stochastics **8**, 145–159 (2004)
10. Brigo, D., Pallavicini, A., Torresetti, R.: Calibration of CDO tranches with the dynamical generalized-Poisson loss model (2007). working paper, Banca IMI
11. Brémaud, P.: Point Processes and Queues: Martingale Dynamics. Springer-Verlag (1981)
12. Cont, R., Deguest, R., Kan, Y.: Default intensities implied by CDO spreads: inversion formula and model calibration (2009). working paper
13. Cont, R., Kan, Y.: Dynamic hedging of portfolio credit derivatives in top-down models (2008). working paper, Columbia University
14. Cont, R., Minca, A.: Recovering portfolio default intensities implied by CDO quotes (2008). working paper, Columbia University
15. Das, S.: Credit Derivatives: CDOs and Structured Credit Products. Wiley (2005)
16. Das, S., Duffie, D., Kapadia, N., Saita, L.: Common failings: how corporate defaults are correlated. Journal of Finance **62**(1), 93–117 (2007)
17. Davis, M., Lo, V.: Infectious defaults. Quantitative Finance **1**, 382–387 (2001)
18. De Koch, J., Kraft, H.: CDOs in chains (2007). working paper, University of Kaiserslautern
19. De Servigny, A., Jobst, N.: The Handbook of Structured Finance. McGraw-Hill (2007)
20. Deguest, R., Kan, Y.H.: A closed-form formula for the local intensity and its application to calibration (2009). working paper
21. Dellacherie, C.: Capacités et processus stochastiques. Springer (1972)
22. Derman, E.: Regimes of volatility (1999). Quantitative Strategies Research Notes, Goldman Sachs
23. Derman, E., Kani, I.: Riding on a smile. RISK **February**, 32–39 (1994)

24. Ding, X., Giesecke, K., Tomecek, P.: Time-changed birth processes and multi-name credit (2006). working paper, Stanford University
25. Dupire, B.: Pricing with a smile. RISK **January**, 18–20 (1994)
26. Eckner, A.: Computational techniques for basic affine models of portfolio credit risk (2007). working paper, Stanford University
27. Elliott, R., Jeanblanc, M., Yor, M.: On models of default risk. Mathematical Finance **10**, 179–196 (2000)
28. Epple, F., Morgan, S., Schloegl, L.: Joint distributions of portfolio losses and exotic portfolio products. International Journal of Theoretical and Applied Finance **10**(4), 733–748 (2007)
29. Errais, E., Giesecke, K., Goldberg, L.: Pricing credit from the top down with affine point processes (2007). working paper, Stanford University
30. Frey, R., Backhaus, J.: Dynamic hedging of synthetic CDO tranches with spread and contagion risk (2007). working paper, University of Leipzig
31. Frey, R., Backhaus, J.: Pricing and hedging of portfolio credit derivatives with interacting default intensities. International Journal of Theoretical and Applied Finance **11**(6), 611–634 (2008)
32. Galiani, S., Kakodkar, A., Shchetkovskiy, M., Jónsson, J.G., Gallo, A.: Base correlation and beyond: interpolation and extrapolation (2006). working paper, credit derivatives strategy, Merrill Lynch
33. Gibson, M.S.: Credit derivatives and risk management (2007). Finance and Economics Discussion Series, Federal Reserve Board, Washington, D.C
34. Giesecke, K., Goldberg, L.: A top down approach to multi-name credit (2006). working paper, Stanford University
35. Herbertsson, A.: Default contagion in large homogeneous portfolios (2007). in The Credit Derivatives Handbook, ed.: G. N. Gregoriou and P. U. Ali., McGraw-Hill
36. Herbertsson, A.: Pricing synthetic CDO tranches in a model with default contagion using the matrix-analytic approach. Journal of Credit Risk **4**(4), 3–35 (2008)
37. Herbertsson, A., Rootzén, H.: Pricing k-th to default swaps under default contagion, the matrix-analytic approach. Journal of Computational Finance **12**(1), 49–78 (2006)
38. Jarrow, R., Yu, F.: Counterparty risk and the pricing of defaultable securities. Journal of Finance **56**, 1765–1799 (2001)
39. Kakodkar, A., Galiani, S., Jónsson, J., Gallo, A.: Credit Derivatives Handbook 2006 - Vol. 2. Merrill Lynch (2006)
40. Karlin, S., Taylor, H.: A First Course in Stochastic Processes. Second Edition, Academic Press (1975)
41. Krekel, M., Partenheimer, J.: The implied loss surface of CDOs (2006). working paper, University of Ulm
42. Kusuoka, S.: A remark on default risk models. Advances in Mathematical Economics **1**, 69–82 (1999)
43. Lando, D.: Credit Risk Modeling and Applications. Princeton University Press (2004)
44. Lando, D., Nielsen, M.S.: Correlation in corporate defaults: contagion or conditional independence (2008). workin paper, Copenhagen Business School
45. Laurent, J.P.: A note on the risk management of CDOs (2006). working paper, ISFA actuarial School, University of Lyon and BNP Paribas
46. Laurent, J.P., Cousin, A., Fermanian, J.D.: Hedging default risk of CDOs in Markovian contagion models (2007). working paper, ISFA, University of Lyon and BNP Paribas
47. Laurent, J.P., Leisen, D.: Building a consistent pricing model from observed option prices (2000). In: Quantitative Analysis in Financial Markets: Collected papers of the New York University Mathematical Finance Seminar, Volume II, 216-238, World Scientific
48. Lopatin, A., Misirpashaev, T.: Two-dimensional Markovian model for dynamics of aggregate credit loss (2007). working paper, NumeriX
49. Meissner, G.: The Definitive Guide to CDOs. Riskbooks (2008)
50. Meyer-Dautrich, S., Wagner, C.: Minimum entropy calibration of CDO tranches (2007). working paper, Unicredit

51. Moler, C., Van Loan, C.: Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. SIAM review **45**(1), 3–49 (2003)
52. Morgan, S., Mortensen, A.: CDO hedging anomalies in the base correlation approach (2007). Lehman Brothers, Quantitative Credit Research Quarterly, October, 49-61
53. Parcell, E., Wood, J.: Wiping the smile off your base (correlation curve) (2007). working paper, Derivative Fitch
54. Parker, E., Brown, M.: The 2003 ISDA credit derivatives definitions (2003). PLC Finance
55. Petrelli, A., Zhang, J., Jobst, N., Kapoor, V.: A practical guide to CDO trading risk management (2007). In: The Handbook of Structured Finance, A. de Servigny and N. Jobst (eds), McGraw Hill, 339-371
56. Protter, P.: Stochastic Integration and Differential Equations. Second Edition, Springer, Berlin (2005)
57. Putyatin, P., Prieul, D., Maslova, S.: A Markovian approach to modelling correlated defaults. RISK **May**, 76–81 (2005)
58. Rubinstein, M.: Implied binomial trees. Journal of Finance **49**(3), 771–818 (1994)
59. Schönbucher, P.: Credit Derivatives Pricing Models. Wiley, Chichester (2003)
60. Schönbucher, P.: Portfolio losses and the term-structure of loss transition rates: a new methodology for the pricing of portfolio credit derivatives (2006). working paper, ETH Zürich
61. Schönbucher, P., Schubert, D.: Copula dependent default risk in intensity models (2001). working paper, Bonn University
62. Torresetti, R., Brigo, D., Pallavicini, A.: Implied expected tranched loss surface from CDO data (2007). working paper, Banca IMI
63. van der Voort, M.: An implied loss model (2006). working paper, ABN Amro and Erasmus University
64. Walker, M.: Simultaneous calibration to a range of portfolio credit derivatives with a dynamic discrete-time multi-step Markov loss model (2007). working paper, University of Toronto
65. Yu, F.: Correlated defaults in intensity-based models. Mathematical Finance **17**(2), 155–173 (2007)