Comparison results for exchangeable credit risk portfolios

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CDO tranches

- Credit portfolio with $n$ reference entities
- $\tau_1, \ldots, \tau_n$ default times
- $(D_{1,t}, \ldots, D_{n,t}) = (1\{\tau_1 \leq t\}, \ldots, 1\{\tau_n \leq t\})$ default indicators at time $t$
- $M_1, \ldots, M_n$ losses given default assumed to be independent of default times
- Aggregate loss:

$$L_t = \sum_{i=1}^{n} M_i D_{i,t}$$

- Which is the impact of dependence on
  - CDO tranche premiums?
  - Risk measures on the aggregate loss associated with the reference portfolio?
De Finetti theorem and factor representation

Homogeneity assumption: default indicators $D_1, \ldots, D_n$ form an exchangeable Bernoulli random vector

**Definition (Exchangeability)**

A random vector $(D_1, \ldots, D_n)$ is exchangeable if its distribution function is invariant for every permutations of its coordinates: $\forall \sigma \in S_n$

$$(D_1, \ldots, D_n) \overset{d}{=} (D_{\sigma(1)}, \ldots, D_{\sigma(n)})$$

Same marginals
De Finetti theorem and factor representation

Assume that \( D_1, \ldots, D_n, \ldots \) is an exchangeable sequence of Bernoulli random variables.

Thanks to de Finetti's theorem, there exists a unique random factor \( \tilde{p} \) such that

\( D_1, \ldots, D_n \) are conditionally independent given \( \tilde{p} \).

Denote by \( F_{\tilde{p}} \) the distribution function of \( \tilde{p} \), then:

\[
P(D_1 = d_1, \ldots, D_n = d_n) = \int_0^1 p^{\sum_i d_i} (1 - p)^{n - \sum_i d_i} F_{\tilde{p}}(dp)
\]

\( \tilde{p} \) is characterized by:

\[
\frac{1}{n} \sum_{i=1}^n D_i \xrightarrow{a.s.} \tilde{p} \quad \text{as} \quad n \to \infty
\]

\( \tilde{p} \) is exactly the loss of the infinitely granular portfolio (Basel 2 terminology).
The **convex order** compares the dispersion level of two random variables:

- **Convex order**: $X \leq_{cx} Y$ if $E[f(X)] \leq E[f(Y)]$ for all convex functions $f$

- **Stop-loss order**: $X \leq_{sl} Y$ if $E[(X - K)^+] \leq E[(Y - K)^+]$ for all $K \in \mathbb{R}$

$$X \leq_{sl} Y \text{ and } E[X] = E[Y] \iff X \leq_{cx} Y$$

- $X \leq_{cx} Y$ if $E[X] = E[Y]$ and $F_X$, the distribution function of $X$, and $F_Y$, the distribution function of $Y$, are such that:

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Comparison results for exchangeable credit risk portfolios
Supermodular order

- The **supermodular order** captures the dependence level among coordinates of a random vector

$$(X_1, \ldots, X_n) \leq_{sm} (Y_1, \ldots, Y_n) \text{ if } E[f(X_1, \ldots, X_n)] \leq E[f(Y_1, \ldots, Y_n)] \text{ for all supermodular functions } f$$

**Definition (Supermodular function)**

A function $f : \mathbb{R}^n \to \mathbb{R}$ is **supermodular** if for all $x \in \mathbb{R}^n$, $1 \leq i < j \leq n$ and $\varepsilon, \delta > 0$ holds

$$f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n)$$

$$\geq f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$$

Müller (1997)

*Stop-loss order for portfolios of dependent risks*

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*) \Rightarrow \sum_{i=1}^{n} M_i D_i \leq_{sl} \sum_{i=1}^{n} M_i D_i^*$$

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Comparison results for exchangeable credit risk portfolios
Main results

Let us compare two credit portfolios with aggregate loss \( L_t = \sum_{i=1}^{n} M_i D_i \) and \( L_t^* = \sum_{i=1}^{n} M_i D_i^* \).

Let \( D_1, \ldots, D_n \) be exchangeable Bernoulli random variables associated with the mixing probability \( \bar{p} \).

Let \( D_1^*, \ldots, D_n^* \) exchangeable Bernoulli random variables associated with the mixing probability \( \bar{p}^* \).

\[ \bar{p} \leq_{cx} \bar{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*) \]

In particular, if \( \bar{p} \leq_{cx} \bar{p}^* \), then:

- \( E[(L_t - a)^+] \leq E[(L_t^* - a)^+] \) for all \( a > 0 \).
- \( \rho(L_t) \leq \rho(L_t^*) \) for all convex risk measures \( \rho \).
Main results

- Let $D_1, \ldots, D_n, \ldots$ be exchangeable Bernoulli random variables associated with the mixing probability $\tilde{p}$
- Let $D_1^*, \ldots, D_n^*, \ldots$ be exchangeable Bernoulli random variables associated with the mixing probability $\tilde{p}^*$

**Theorem (reverse implication)**

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*), \forall n \in \mathbb{N} \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^*.$$
1 Comparison results
   • De Finetti theorem and factor representation
   • Stochastic orders
   • Main results

2 Application to several popular CDO pricing models
   • Factor copula approaches
   • Multivariate Poisson model
   • Structural model
Analysis of the dependence structure in several popular CDO pricing models

An increase of the dependence parameter leads to:

- a decrease of $[0\%, b]$ equity tranche premiums (which guarantees the uniqueness of the market base correlation)
- an increase of $[a, 100\%]$ senior tranche premiums
Additive factor copula approaches

- The dependence structure of default times is described by some latent variables $V_1, \ldots, V_n$ such that:
  - $V_i = \rho V + \sqrt{1 - \rho^2} \bar{V}_i, \ i = 1 \ldots n$
  - $V, \bar{V}_i, \ i = 1 \ldots n$ independent
  - $\tau_i = G^{-1}(H_\rho(V_i)), \ i = 1 \ldots n$
    - $G$: distribution function of $\tau_i$
    - $H_\rho$: distribution function of $V_i$
  - $D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n$ are conditionally independent given $V$
  - $\frac{1}{n} \sum_{i=1}^{n} D_i \overset{a.s.}{\rightarrow} E[D_i \mid V] = P(\tau_i \leq t \mid V) = \bar{p}$
Theorem

For any fixed time horizon $t$, denote by $D_i = 1_{\{\tau_i \leq t\}}, \ i = 1 \ldots n$ and $D_i^* = 1_{\{\tau_i^* \leq t\}}, \ i = 1 \ldots n$ the default indicators corresponding to (resp.) $\rho$ and $\rho^*$, then:

$$\rho \leq \rho^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

This framework includes popular factor copula models:

- One factor Gaussian copula - the industry standard for the pricing of CDO tranches
- Double t: Hull and White(2004)
- NIG, double NIG: Guegan and Houdain(2005), Kalemanova, Schmid and Werner(2007)
- Double Variance Gamma: Moosbrucker(2006)

- $V$ is a positive random variable with Laplace transform $\varphi^{-1}$
- $U_1, \ldots, U_n$ are independent Uniform random variables independent of $V$
- $V_i = \varphi^{-1} \left( -\frac{\ln U_i}{V} \right)$, $i = 1 \ldots n$ (Marshall and Olkin (1988))
  - $(V_1, \ldots, V_n)$ follows a $\varphi$-archimedean copula
  - $P(V_1 \leq v_1, \ldots, V_n \leq v_n) = \varphi^{-1}(\varphi(v_1) + \ldots + \varphi(v_n))$
- $\tau_i = G^{-1}(V_i)$
  - $G$: distribution function of $\tau_i$
- $D_i = 1\{\tau_i \leq t\}$, $i = 1 \ldots n$ independent knowing $V$
- $\frac{1}{n} \sum_{i=1}^{n} D_i \overset{a.s.}{\rightarrow} E[D_i \mid V] = P(\tau_i \leq t \mid V)$
Archimedean copula

- Conditional default probability: \( \tilde{p} = \exp \{-\varphi(G(t)V)\} \)

<table>
<thead>
<tr>
<th>Copula</th>
<th>Generator ( \varphi )</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clayton</td>
<td>( t^{-\theta} - 1 )</td>
<td>( \theta \geq 0 )</td>
</tr>
<tr>
<td>Gumbel</td>
<td>( (-\ln(t))^\theta )</td>
<td>( \theta \geq 1 )</td>
</tr>
<tr>
<td>Franck</td>
<td>(-\ln \left[ \frac{(1 - e^{-\theta t})}{(1 - e^{-\theta})} \right] )</td>
<td>( \theta \in \mathbb{R}^* )</td>
</tr>
</tbody>
</table>

Theorem

\( \theta \leq \theta^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*) \)
Archimedean copula

- Clayton copula
- Mixture distributions are ordered with respect to the convex order
Multivariate Poisson model


- $\bar{N}^i_t$ Poisson with parameter $\bar{\lambda}$: idiosyncratic risk
- $N^i_t$ Poisson with parameter $\lambda$: systematic risk
- $(B^i_j)_{i,j}$ Bernoulli random variable with parameter $p$
- All sources of risk are independent
- $N^i_t = \bar{N}^i_t + \sum_{j=1}^{N^i_t} B^i_j, \ i = 1 \ldots n$
- $\tau^i = \inf\{t > 0|N^i_t > 0\}, \ i = 1 \ldots n$
Multivariate Poisson model

- Dependence structure of \((\tau_1, \ldots, \tau_n)\) is the Marshall-Olkin copula
- \(\tau_i \sim \text{Exp}(\bar{\lambda} + p\lambda)\)
- \(D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n\) are conditionally independent given \(N_t\)
- \(\frac{1}{n} \sum_{i=1}^{n} D_i \overset{a.s.}{\longrightarrow} E[D_i \mid N_t] = P(\tau_i \leq t \mid N_t)\)
- Conditional default probability:

\[
\tilde{p} = 1 - (1 - p)^{N_t} \exp(-\bar{\lambda}t)
\]
Comparison of two multivariate Poisson models with parameter sets 
\((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\)

Supermodular order comparison requires equality of marginals:
\[ \bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda^* \]

3 comparison directions:
- \(p = p^*\): \(\bar{\lambda}\) v.s \(\lambda\)
- \(\lambda = \lambda^*\): \(\bar{\lambda}\) v.s \(p\)
- \(\bar{\lambda} = \bar{\lambda}^*\): \(\lambda\) v.s \(p\)
Theorem ($p = p^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p\lambda^*$, then:

$$\lambda \leq \lambda^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- Computation of $E[(L_t - a)^+]$:
  - 30 names
  - $M_i = 1, \; i = 1 \ldots n$
- When $\lambda$ increases, the aggregate loss increases with respect to stop-loss order
Theorem \((\lambda = \lambda^*)\)

Let parameter sets \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\) be such that \(\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda\), then:

\[
p \leq p^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)
\]

- Convex order for mixture probabilities
Theorem ($\lambda = \lambda^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda$, then:

$$p \leq p^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq c_x \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- Computation of $E[(L_t - K)^+]$:
  - 30 names
  - $M_i = 1, \ i = 1 \ldots n$
- When $p$ increases, the aggregate loss increases with respect to stop-loss order
Comparison results
Application to several popular CDO pricing models
Multivariate Poisson model
Structure model

Multivariate Poisson model

**Theorem \((\bar{\lambda} = \bar{\lambda}^*)\)**

*Let parameter sets \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\) be such that \(p\lambda = p^*\lambda^*\), then:

\[
p \leq p^*, \ \lambda \geq \lambda^* \Rightarrow \bar{p} \leq_{cx} \bar{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)
\]

- Computation of \(E[(L_t - K)^+]\):
  - 30 names
  - \(M_i = 1, \ i = 1 \ldots n\)
- When \(p\) increases, the aggregate loss increases with respect to stop-loss order
Hull, Predescu and White (2005)

- Consider $n$ firms
- Let $V_{i,t}, i = 1 \ldots n$ be their asset dynamics
  \[ V_{i,t} = \rho V_t + \sqrt{1 - \rho^2} \bar{V}_{i,t}, \quad i = 1 \ldots n \]
- $V, \bar{V}_i, i = 1 \ldots n$ are independent standard Wiener processes
- Default times as first passage times:
  \[ \tau_i = \inf \{ t \in \mathbb{R}^+ | V_{i,t} \leq f(t) \}, \quad i = 1 \ldots n, \quad f : \mathbb{R} \rightarrow \mathbb{R} \text{ continuous} \]
- $D_i = 1_{\{\tau_i \leq \tau\}}, i = 1 \ldots n$ are conditionally independent given $\sigma(V_t, t \in [0, T])$
Theorem

For any fixed time horizon $T$, denote by $D_i = 1\{\tau_i \leq T\}$, $i = 1 \ldots n$ and $D_i^* = 1\{\tau_i^* \leq T\}$, $i = 1 \ldots n$ the default indicators corresponding to (resp.) $\rho$ and $\rho^*$, then:

$$\rho \leq \rho^* \Rightarrow (D_1, \ldots, D_n) \leq_{\text{sm}} (D_1^*, \ldots, D_n^*)$$
Structural model

- $\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} \tilde{p}$
- $\frac{1}{n} \sum_{i=1}^{n} D_i^* \xrightarrow{a.s.} \tilde{p}^*$

Empirically, mixture probabilities are ordered with respect to the convex order: $\tilde{p} \leq_{cx} \tilde{p}^*$
Conclusion

- When considering an exchangeable vector of default indicators, the conditional independence assumption is not restrictive thanks to de Finetti’s theorem.

- The mixing probability (the factor) can be viewed as the loss of an infinitely granular portfolio.

- We completely characterize the supermodular order between exchangeable default indicator vectors in term of the convex ordering of corresponding mixing probabilities.

- We show that the mixing probability is the key input to study the impact of dependence on CDO tranche premiums.

- Comparison analysis can be performed with the same method within a large class of CDO pricing models.
Exchangeability: how realistic is a homogeneous assumption?

- Homogeneity of default marginals is an issue when considering the pricing and the hedging of CDO tranches
  - ex: Sudden surge of GMAC spreads in the CDX index in May, 2005
  - This event dramatically impacts the equity tranche compared to others tranches
- But composition of standard indices are updated every semester, resulting in an increase of portfolio homogeneity
- It may be reasonable to split a credit portfolio in several homogeneous sub-portfolios (by economic sectors for example)
  - Then, for each sub-portfolio, we can find a specific factor and apply the previous comparison analysis
  - The initial credit portfolio can thus be associated with a vector of factors (one by sector)
  - Is it possible to relate comparison between global aggregate losses to comparison between vectors of random factors?
Are comparisons in a static framework restrictive?

- Are comparisons among aggregate losses at fixed horizons too restrictive?
- Computation of CDO tranche premiums only requires marginal loss distributions at several horizons
  - Comparison among aggregate losses at different dates is sufficient
- However, comparison of more complex products such as options on tranche or forward started CDOs are not possible in this framework
- Building a framework in which one can compare directly aggregate loss processes is a subject of future research