Comparison results for credit risk portfolios

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Introduction

- Presentation devoted to risk analysis of credit portfolios
- In credit risk portfolio modelling, dependence among default events is a crucial assumption
- We will investigate tranches of Collateralized Debt Obligation (CDO)
- Which is the impact of the dependence on
  - CDO tranche premiums?
  - Risk measures on the aggregate loss?
CDO tranches

- Slice the credit portfolio into different risk levels or CDO tranches
- ex: CDO tranche on standardized Index such as CDX North America or Itraxx Europe

- [0, 3%] equity tranche is subordinated to [3, 6%] junior mezzanine tranche
- [3, 6%] junior mezzanine tranche is subordinated to [6, 9%] mezzanine tranche and so on,...
CDO tranches

- Each CDO tranche is a bilateral contract between a buyer of protection and a seller of protection:

  ![Diagram]

  - Buyer of Protection
  - Seller of Protection
  - Quarterly premium payments
  - Payment when defaults affect the tranche

- CDO tranche cash flows are driven by the aggregate loss process
CDO tranches

- Credit portfolio with \( n \) reference entities
- \( \tau_1, \ldots, \tau_n \) default times
- \((D_1, \ldots, D_n) = (1_{\tau_1 \leq t}, \ldots, 1_{\tau_n \leq t})\) default indicators at time \( t \)
- \( M_1, \ldots, M_n \) losses given default assumed to be independent of default times
- Aggregate loss:

\[
L_t = \sum_{i=1}^{n} M_i 1_{\tau_i \leq t}
\]

- Dynamics of the aggregate loss process:
CDO tranches

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Dynamics of the aggregate loss process:
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  \[
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  \]
- Dynamics of the aggregate loss process:
$L_t^{(a,b)}$ has a call spread payoff with respect to the aggregate loss:

\[
L_t^{(a,b)} = (L_t - a)^+ - (L_t - b)^+
\]

Loss on CDO tranche $[a, b]$:

\[
L_t^{(a,b)} = (L_t - a)^+ - (L_t - b)^+
\]

Tranche premiums only involves call options on the aggregate loss $L_t$:

\[
E [(L_t - a)^+] - E [(L_t - b)^+]
\]
Motivation

- Specify the dependence structure of default indicators $D_1, \ldots, D_n$ which leads to:
  - an increase of the value of call options $E \left[ (L_t - a)^+ \right]$ for all strike level $a > 0$
  - an increase of convex risk measures on $L_t$ (TVaR, Wang risk measures)
- Comparison between homogeneous credit portfolios
  - $D_1, \ldots, D_n$ are assumed to be exchangeable Bernoulli random variables
  - De Finetti Theorem leads to a factor representation
- Application to several default risk models
Homogeneity assumption: default indicators $D_1, \ldots, D_n$ forms an exchangeable Bernoulli random vector.

**Definition (Exchangeability)**

A random vector $(D_1, \ldots, D_n)$ is exchangeable if its distribution function is invariant for every permutations of its coordinates: $\forall \sigma \in S_n$

$$(D_1, \ldots, D_n) \overset{d}{=} (D_{\sigma(1)}, \ldots, D_{\sigma(n)})$$
De Finetti theorem and factor representation

- Assume that $D_1, \ldots, D_n, \ldots$ is an exchangeable sequence of Bernoulli random variables.
- Thanks to de Finetti theorem, there exists a random factor $\tilde{p}$ such that $D_1, \ldots, D_n$ are conditionally independent given $\tilde{p}$.
- Denote by $F_{\tilde{p}}$ the distribution function of $\tilde{p}$, then:

$$P(D_1 = d_1, \ldots, D_n = d_n) = \int_0^1 p^{\sum d_i} (1 - p)^{n - \sum d_i} F_{\tilde{p}}(dp)$$

- $\tilde{p}$ is characterized by:

$$\frac{1}{n} \sum_{i=1}^n D_i \overset{a.s.}{\rightarrow} \tilde{p} \quad \text{as} \quad n \rightarrow \infty$$
The convex order compares the dispersion level of two random variables.

\[ X \leq_{cx} Y \text{ if } E[f(X)] \leq E[f(Y)] \text{ for all convex functions } f \]

Particularly, if \( X \leq_{cx} Y \) then \( E[X] = E[Y] \) and \( \text{Var}(X) \leq \text{Var}(Y) \).

Two important consequences of the convex order:

- If \( X \leq_{cx} Y \) then \( E[(X - a)^+] \leq E[(Y - a)^+] \) for all \( a > 0 \).
- If \( X \leq_{cx} Y \) then \( \rho(X) \leq \rho(Y) \) for all law invariant and convex risk measures \( \rho \) (Bäuerle and Müller(2005)).
Supermodular order

- The supermodular order captures the dependence level among coordinates of a random vector
- \((X_1, \ldots, X_n) \leq_{sm} (Y_1, \ldots, Y_n)\) if \(E[f(X_1, \ldots, X_n)] \leq E[f(Y_1, \ldots, Y_n)]\) for all supermodular function \(f\)

**Definition (Supermodular function)**

A function \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) is **supermodular** if for all \(x \in \mathbb{R}^n\), \(1 \leq i < j \leq n\) and \(\varepsilon, \delta > 0\) holds

\[
f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n) \\
\geq f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)
\]

- Consequences of new defaults are always worse when other defaults have already occurred
- If \((D_1, \ldots, D_n) \leq_{sm} (D_1, \ldots, D_n)\) then \(\sum_{i=1}^{n} M_i D_i \leq_{cx} \sum_{i=1}^{n} M_i D_i\) (Müller(1997))
Main results

- Let us compare two credit portfolios with aggregate loss $L_t = \sum_{i=1}^{n} M_i D_i$ and $L_t^* = \sum_{i=1}^{n} M_i D_i^*$.
- Let $D_1, \ldots, D_n$ be exchangeable Bernoulli random variables associated with the mixture factor $\tilde{p}$.
- $D_1^*, \ldots, D_n^*$ exchangeable Bernoulli random variables associated with the mixture factor $\tilde{p}^*$.

**Theorem**

$$\tilde{p} \preceq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \preceq_{sm} (D_1^*, \ldots, D_n^*)$$

$$\Rightarrow E[(L_t - a)^+] \leq E[(L_t^* - a)^+] \text{ for all } a > 0$$

$$\Rightarrow \rho(L_t) \leq \rho(L_t^*) \text{ for all convex risk measures } \rho$$

**Theorem**

$$(D_1, \ldots, D_n) \preceq_{sm} (D_1^*, \ldots, D_n^*), \forall n \in \mathbb{N} \Rightarrow \tilde{p} \preceq_{cx} \tilde{p}^* \quad (1)$$
Additive factor copula approaches

- The dependence structure of default times is described by some latent variables $V_1, \ldots, V_n$ such that:
  - $V_i = \rho V + \sqrt{1 - \rho^2} \bar{V}_i, \ i = 1 \ldots n$
  - $V, \bar{V}_i, \ i = 1 \ldots n$ independent
  - $\tau_i = G^{-1}(H_\rho(V_i)), \ i = 1 \ldots n$
    - $G$: distribution function of $\tau_i$
    - $H_\rho$: distribution function of $V_i$
  - $D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n$ are conditionally independent given $V$
  - $\frac{1}{n} \sum_{i=1}^{n} D_i \rightarrow E[D_i \mid V] = P(\tau_i \leq t \mid V) = \tilde{p}$
Additive factor copula approaches

Theorem

For any fixed time horizon $t$, denote by $D_i = 1\{\tau_i \leq t\}$, $i = 1 \ldots n$ and $D_i^* = 1\{\tau_i^* \leq t\}$, $i = 1 \ldots n$ the default indicators corresponding to (resp.) $\rho$ and $\rho^*$, then:

$$\rho \leq \rho^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- This framework includes popular factor copula models:
  - One factor Gaussian copula - the industry standard for the pricing of CDO tranches
  - Double t: Hull and White(2004)
  - NIG, double NIG: Guegan and Houdain(2005), Kalemanova, Schmid and Werner(2005)
  - Double Variance Gamma: Moosbrucker(2005)
Hull, Predescu and White (2005)

- Consider \( n \) firms
- Let \( X^i_t, \ i = 1 \ldots n \) be their asset dynamics
  \[
  X^i_t = \rho W_t + \sqrt{1 - \rho^2} W^i_t, \ i = 1 \ldots n
  \]
- \( W, W^i, \ i = 1 \ldots n \) are independent standard Wiener processes
- Default times as first passage times:
  \[
  \tau_i = \inf \{ t \in \mathbb{R}^+ | X^i_t \leq f(t) \}, \ i = 1 \ldots n,\ f : \mathbb{R} \to \mathbb{R} \text{ continuous}
  \]
- \( D_i = 1_{\{\tau_i \leq T\}}, \ i = 1 \ldots n \) are conditionally independent given \( \sigma(W_t, t \in [0, T]) \)
Theorem

For any fixed time horizon $T$, denote by $D_i = 1_{\{\tau_i \leq T\}}$, $i = 1 \ldots n$ and $D_i^* = 1_{\{\tau_i^* \leq T\}}$, $i = 1 \ldots n$ the default indicators corresponding to (resp.) $\rho$ and $\rho^*$, then:

$$\rho \leq \rho^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$
Archimedean copula

<table>
<thead>
<tr>
<th>Copula name</th>
<th>Generator $\varphi$</th>
<th>$V$-distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clayton</td>
<td>$t^{-\theta} - 1$</td>
<td>Gamma$(1/\theta)$</td>
</tr>
<tr>
<td>Gumbel</td>
<td>$(-\ln(t))^{\theta}$</td>
<td>$\alpha$-Stable, $\alpha = 1/\theta$</td>
</tr>
<tr>
<td>Frank</td>
<td>$-\ln \left[ (1 - e^{-\theta t})/(1 - e^{-\theta}) \right]$</td>
<td>Logarithmic series</td>
</tr>
</tbody>
</table>

Theorem

$\theta \leq \theta^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$

- Other comparison results for multivariate Poisson models
When considering homogeneous credit portfolios, the factor representation of default indicators is not restrictive.

- Thanks to De Finetti’s theorem, there exists a mixture probability $\tilde{p}$ such that default indicators are conditionally independent given $\tilde{p}$

This mixture probability is the key input to analyze the impact of dependence on:

- CDO tranche premiums
- Convex risk measures on the aggregate loss

This analysis can be performed for several popular default risk models