Comparison results for homogenous credit portfolios

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1 Comparison of Exchangeable Bernoulli random vectors
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Exchangeability assumption

- $n$ defaultable firms
- $\tau_1, \ldots, \tau_n$ default times
- $(D_1, \ldots, D_n) = (1\{\tau_1 \leq t\}, \ldots, 1\{\tau_n \leq t\})$ default indicators
- Homogeneity assumption: default dates are assumed to be exchangeable

**Definition (Exchangeability)**

A random vector $(\tau_1, \ldots, \tau_n)$ is exchangeable if its distribution function is invariant by permutation: $\forall \sigma \in S_n$

$$(\tau_1, \ldots, \tau_n) \overset{d}{=} (\tau_{\sigma(1)}, \ldots, \tau_{\sigma(n)})$$

- Same marginals
Suppose that $D_1, \ldots, D_n, \ldots$ is an exchangeable sequence of Bernoulli random variables

There exists a random factor $\tilde{p}$ such that

$D_1, \ldots, D_n$ are independent knowing $\tilde{p}$

Denote by $F_{\tilde{p}}$ the distribution function of $\tilde{p}$, then:

$$P(D_1 = d_1, \ldots, D_n = d_n) = \int_{0}^{1} p^{\sum d_i} (1 - p)^{n - \sum d_i} F_{\tilde{p}}(dp)$$

$\tilde{p}$ is characterized by:

$$\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} \tilde{p} \quad \text{as} \quad n \to \infty$$
Stochastic orders

- $X \preceq_{cx} Y$ if $E[f(X)] \leq E[f(Y)]$ for all convex functions $f$
- $X \preceq_{sl} Y$ if $E[(X - K)^+] \leq E[(Y - K)^+]$ for all $K \in \mathbb{R}$
  - $X \preceq_{sl} Y$ and $E[X] = E[Y] \iff X \preceq_{cx} Y$
- $X \preceq_{sm} Y$ if $E[f(X)] \leq E[f(Y)]$ for all supermodular functions $f$

**Definition (Supermodular function)**

A function $f : \mathbb{R}^n \to \mathbb{R}$ is **supermodular** if for all $x \in \mathbb{R}^n$, $1 \leq i < j \leq n$ and $\varepsilon, \delta > 0$ holds

$$f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n) \geq f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$$

- consequences of new defaults are always worse when other defaults have already occurred
(\(D_1, \ldots, D_n\)) and (\(D_1^*, \ldots, D_n^*\)) two exchangeable default indicator vectors

- \(M_i\) loss given default
- Aggregate losses:

\[
L_t = \sum_{i=1}^{n} M_i D_i \\
L_t^* = \sum_{i=1}^{n} M_i D_i^*
\]

Müller(1997)

Stop-loss order for portfolios of dependent risks.

\((D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*) \Rightarrow L_t \leq_{sl} L_t^*\)
Theorem

Let $\mathbf{D} = (D_1, \ldots, D_n)$ and $\mathbf{D}^* = (D_1^*, \ldots, D_n^*)$ be two exchangeable Bernoulli random vectors with (resp.) $F$ and $F^*$ as mixture distributions. Then:

$$F \leq_{cx} F^* \Rightarrow \mathbf{D} \leq_{sm} \mathbf{D}^*$$

and

Theorem

Let $D_1, \ldots, D_n, \ldots$ and $D_1^*, \ldots, D_n^*, \ldots$ be two exchangeable sequences of Bernoulli random variables. We denote by $F$ (resp. $F^*$) the distribution function associated with the mixing measure. Then,

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*), \quad \forall n \in \mathbb{N} \Rightarrow F \leq_{cx} F^*.$$
Multivariate Poisson model

- $\tilde{N}_t^i$ Poisson with parameter $\bar{\lambda}$: idiosyncratic risk
- $N_t$ Poisson with parameter $\lambda$: systematic risk
- $(B_j^i)_{i,j}$ Bernoulli random variable with parameter $p$
- All sources of risk are independent
- $N_t^i = \tilde{N}_t^i + \sum_{j=1}^{N_t} B_j^i$, $i = 1 \ldots n$
- $\tau_i = \inf\{t > 0 | N_t^i > 0\}$, $i = 1 \ldots n$

Multivariate Poisson model

- $\tau_i \sim \text{Exp}(\bar{\lambda} + p\lambda)$
- $D_i = 1{\{\tau_i \leq t\}}, \ i = 1 \ldots n$ are independent knowing $N_t$
- $\frac{1}{n} \sum_{i=1}^{n} D_i \overset{a.s.}{\rightarrow} E[D_i \mid N_t] = P(\tau_i \leq t \mid N_t)$
- Conditional default probability:
  $$\tilde{p} = 1 - (1 - p)^{N_t} \exp(-\bar{\lambda}t)$$
Multivariate Poisson model

- Comparison of two multivariate Poisson models with parameter sets 
  \((\tilde{\lambda}, \lambda, p)\) and \((\tilde{\lambda}^*, \lambda^*, p^*)\)

- Supermodular order comparison requires equality of marginals:
  \(\tilde{\lambda} + p\lambda = \tilde{\lambda}^* + p^*\lambda^*\)

- Comparison directions:
  - \(p = p^*\): \(\tilde{\lambda}\) v.s \(\lambda\)
  - \(\lambda = \lambda^*\): \(\tilde{\lambda}\) v.s \(p\)
Theorem \((\rho = \rho^*)\)

Let parameter sets \((\bar{\lambda}, \lambda, \rho)\) and \((\bar{\lambda}^*, \lambda^*, \rho^*)\) be such that \(\bar{\lambda} + \rho \lambda = \bar{\lambda}^* + \rho \lambda^*\), then:

\[
\lambda \leq \lambda^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{\rho} \leq_{\text{cx}} \tilde{\rho}^* \Rightarrow (D_1, \ldots, D_n) \leq_{\text{sm}} (D_1^*, \ldots, D_n^*)
\]
Comparison of Exchangeable Bernoulli random vectors

Application to Credit Risk Management

Conclusion

Multivariate Poisson model

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Comparison results for homogenous credit portfolios

Theorem ($\lambda = \lambda^*$)

Let parameter sets ($\bar{\lambda}, \lambda, p$) and ($\bar{\lambda}^*, \lambda^*, p^*$) be such that $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda$, then:

$$p \leq p^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$
Structural Model

Hull, Predescu and White (2005)

- Consider $n$ firms
- Let $X_t^i, \ i = 1 \ldots n$ be their asset dynamics
  \[ X_t^i = \rho W_t + \sqrt{1 - \rho^2} W_t^i, \ i = 1 \ldots n \]
- $W, W^i, \ i = 1 \ldots n$ are independent standard Wiener processes
- Default times as first passage times:
  \[ \tau_i = \inf\{t \in \mathbb{R}^+ | X_t^i \leq f(t)\}, \ i = 1 \ldots n, \ f : \mathbb{R} \to \mathbb{R} \text{ continuous} \]
- $D_i = 1\{\tau_i \leq \tau\}, \ i = 1 \ldots n$ are independent knowing $\sigma(W_t, t \in [0, T])$
- \[ \frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} \tilde{p} \]
**Theorem**

For any fixed time horizon $T$, denote by $D_i = 1\{\tau_i \leq T\}$, $i = 1 \ldots n$ and $D_i^* = 1\{\tau_i^* \leq T\}$, $i = 1 \ldots n$ the default indicators corresponding to (resp.) $\rho$ and $\rho^*$, then:

$$\rho \leq \rho^* \Rightarrow (D_1, \ldots, D_n) \preceq_{sm} (D_1^*, \ldots, D_n^*)$$
Archimedean copula

<table>
<thead>
<tr>
<th>Copula name</th>
<th>Generator $\varphi$</th>
<th>$V$-distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clayton</td>
<td>$t^{-\theta} - 1$</td>
<td>Gamma($1/\theta$)</td>
</tr>
<tr>
<td>Gumbel</td>
<td>$(-\ln(t))^\theta$</td>
<td>$\alpha$-Stable, $\alpha = 1/\theta$</td>
</tr>
<tr>
<td>Franck</td>
<td>$-\ln \left( \frac{(1 - e^{-\theta t})/(1 - e^{-\theta})}{(1 - e^{-\theta})} \right)$</td>
<td>Logarithmic series</td>
</tr>
</tbody>
</table>

Theorem

$\alpha \leq \alpha^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$

$\tilde{p}(\theta) \leq_{cx} \tilde{p}(\theta^*)$
Additive copula framework

- \( V_i = \rho V + \sqrt{1 - \rho^2} \bar{V}_i \)
- \( V, V_i \ i = 1 \ldots n \) independent
- Laws of \( V, V_i \ i = 1 \ldots n \) do not depend on the dependence parameter \( \rho \)
- Standard copula models:
  - Gaussian, Student \( t \)
  - Double \( t \): Hull and White(2004)
  - NIG, double NIG: Guegan and Houdain(2005), Kalemanova, Schmid and Werner(2005)
  - Double Variance Gamma: Moosbrucker(2005)

**Theorem**

\[
\rho \leq \rho^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)
\]
Conclusion

- Characterization of supermodular order for exchangeable Bernoulli random vectors
- Comparison of CDO tranche premiums in several pricing models
- Unified way of presenting default risk models