Comparison results for credit risk portfolios

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In credit risk portfolio modelling, dependence among default events is a crucial assumption. We will investigate tranches of Collateralized Debt Obligation (CDO), which is the impact of dependence on CDO tranche premiums? Risk measures on the aggregate loss associated with the reference portfolio?
Contents

1 CDO tranches

2 Comparison results
   - Motivation
   - De Finetti theorem and factor representation
   - Stochastic orders
   - Main results

3 Application to several popular CDO pricing models
   - Factor copula approaches
   - Structural model
   - Multivariate Poisson model
Slice the credit portfolio into different risk levels or **CDO tranches**. 

- CDO tranche on **standardized Index** such as CDX North America or Itraxx Europe.

- [0, 3%] equity tranche is subordinated to [3, 6%] junior mezzanine tranche.
- [3, 6%] junior mezzanine tranche is subordinated to [6, 9%] mezzanine tranche and so on, ...
Credit portfolio with \( n \) reference entities

\( \tau_1, \ldots, \tau_n \) default times

\( (D_1, \ldots, D_n) = (1_{\{\tau_1 \leq t\}}, \ldots, 1_{\{\tau_n \leq t\}}) \) default indicators at time \( t \)

\( M_1, \ldots, M_n \) losses given default assumed to be independent of default times

Aggregate loss:

\[
L_t = \sum_{i=1}^{n} M_i 1_{\{\tau_i \leq t\}}
\]

Dynamics of Losses \( L_t^{[a,b]} \) affecting CDO tranche \([a, b] \):
CDO tranches

- $L_t^{[a,b]}$ has a call spread payoff with respect to the aggregate loss:

\[
L_t^{[a,b]} = (L_t - a)^+ - (L_t - b)^+
\]

- Loss on CDO tranche $[a, b]$:

\[
L_t^{[a,b]} = (L_t - a)^+ - (L_t - b)^+
\]

- Computation of CDO Tranche premiums only involves call options on the aggregate loss $L_t$:

\[
E [(L_t - a)^+] - E [(L_t - b)^+]
\]

- for different time horizons $t$
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Motivation

- Specify the dependence structure of default indicators $D_1, \ldots, D_n$ which leads to:
  - an increase of the value of call options $E \left[ (L_t - a)^+ \right]$ for all strike level $a > 0$
  - an increase of convex risk measures on $L_t$ (TVaR, Wang risk measures)

- Comparison between homogeneous credit portfolios
  - $D_1, \ldots, D_n$ are assumed to be exchangeable Bernoulli random variables
  - De Finetti’s theorem leads to a factor representation of $D_1, \ldots, D_n$

- Application to several popular CDO pricing models
De Finetti theorem and factor representation

- Homogeneity assumption: default indicators $D_1, \ldots, D_n$ forms an exchangeable Bernoulli random vector

**Definition (Exchangeability)**

A random vector $(D_1, \ldots, D_n)$ is exchangeable if its distribution function is invariant for every permutations of its coordinates: $\forall \sigma \in S_n$

$$(D_1, \ldots, D_n) \overset{d}{=} (D_{\sigma(1)}, \ldots, D_{\sigma(n)})$$

- Same marginals
Assume that $D_1, \ldots, D_n, \ldots$ is an exchangeable sequence of Bernoulli random variables.

Thanks to de Finetti’s theorem, there exists a random factor $\tilde{p}$ such that $D_1, \ldots, D_n$ are conditionally independent given $\tilde{p}$.

Denote by $F_{\tilde{p}}$ the distribution function of $\tilde{p}$, then:

$$P(D_1 = d_1, \ldots, D_n = d_n) = \int_0^1 p^{\sum i d_i} (1 - p)^{n - \sum i d_i} F_{\tilde{p}}(dp)$$

Finite exchangeability only leads to a sign measure Jaynes (1986)

$\tilde{p}$ is characterized by:

$$\frac{1}{n} \sum_{i=1}^n D_i \xrightarrow{a.s.} \tilde{p} \quad \text{as} \quad n \to \infty$$

$\tilde{p}$ is exactly the loss of the infinitely granular portfolio (Bâle 2 terminology)
Stochastic orders

- The convex order compares the **dispersion level** of two random variables.
- **Convex order**: \( X \leq_{cx} Y \) if \( E[f(X)] \leq E[f(Y)] \) for all convex functions \( f \).
- **Stop-loss order**: \( X \leq_{sl} Y \) if \( E[(X - K)^+] \leq E[(Y - K)^+] \) for all \( K \in \mathbb{R} \)
  - \( X \leq_{sl} Y \) and \( E[X] = E[Y] \iff X \leq_{cx} Y \)
- \( X \leq_{cx} Y \) if \( E[X] = E[Y] \) and \( F_X \), the distribution function of \( X \) and \( F_Y \), the distribution function of \( Y \) are such that:

![Diagram showing stochastic orders](image)
Supermodular order

- The supermodular order captures the dependence level among coordinates of a random vector
- \((X_1, \ldots, X_n) \leq_{sm} (Y_1, \ldots, Y_n)\) if \(E[f(X_1, \ldots, X_n)] \leq E[f(Y_1, \ldots, Y_n)]\) for all supermodular function \(f\)

**Definition (Supermodular function)**

A function \(f: \mathbb{R}^n \rightarrow \mathbb{R}\) is **supermodular** if for all \(x \in \mathbb{R}^n, 1 \leq i < j \leq n\) and \(\varepsilon, \delta > 0\) holds

\[
f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n) \\
\geq f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)
\]

- Consequences of new defaults are always worse when other defaults have already occurred
Review of literature

Müller (1997)
Stop-loss order for portfolios of dependent risks

\[(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*) \Rightarrow \sum_{i=1}^{n} M_i D_i \leq_{sl} \sum_{i=1}^{n} M_i D_i^*\]

Bäuerle and Müller (2005)
Stochastic orders ans risk measures: Consistency and bounds

\[X \leq_{sl} Y \Rightarrow \rho(X) \leq \rho(Y)\]

for all law-invariant, convex risk measures \(\rho\)

Lefèvre and Utev (1996)
Comparing sums of exchangeable Bernoulli random variables

\[\tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow \sum_{i=1}^{n} D_i \leq_{sl} \sum_{i=1}^{n} D_i^*\]
Main results

- Let us compare two credit portfolios with aggregate loss $L_t = \sum_{i=1}^{n} M_i D_i$ and $L_t^* = \sum_{i=1}^{n} M_i D_i^*$.
- Let $D_1, \ldots, D_n$ be exchangeable Bernoulli random variables associated with the mixture probability $\tilde{p}$.
- Let $D_1^*, \ldots, D_n^*$ exchangeable Bernoulli random variables associated with the mixture probability $\tilde{p}^*$.

**Theorem**

$$\tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- In particular, if $\tilde{p} \leq_{cx} \tilde{p}^*$, then:
  - $E[(L_t - a)^+] \leq E[(L_t^* - a)^+]$ for all $a > 0$.
  - $\rho(L_t) \leq \rho(L_t^*)$ for all convex risk measures $\rho$. 
Main results

- Let $D_1, \ldots, D_n, \ldots$ be exchangeable Bernoulli random variables associated with the mixture probability $\tilde{p}$
- Let $D_1^*, \ldots, D_n^*, \ldots$ be exchangeable Bernoulli random variables associated with the mixture probability $\tilde{p}^*$

Theorem (reverse implication)

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*), \forall n \in \mathbb{N} \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^*.$$
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Comparison results for credit risk portfolios
Ordering of CDO tranche premiums

Burtschell, Gregory, and Laurent (2008)

A comparative analysis of CDO pricing models

- Analysis of the dependence structure within some factor copula models such as:
  - Gaussian, Student $t$, Double $t$, Clayton, Marshall-Olkin copula
- An increase of the dependence parameter leads to:
  - a decrease of $[0\%, b]$ equity tranches premiums (which guaranties the uniqueness of the market base correlation)
  - an increase of $[a, 100\%]$ senior tranches premiums
Additive factor copula approaches

- The dependence structure of default times is described by some latent variables $V_1, \ldots, V_n$ such that:
  \[ V_i = \rho V + \sqrt{1 - \rho^2} \bar{V}_i, \quad i = 1 \ldots n \]
- $V, \bar{V}_i, \ i = 1 \ldots n$ independent
- $\tau_i = G^{-1}(H_\rho(V_i)), \ i = 1 \ldots n$
  - $G$: distribution function of $\tau_i$
  - $H_\rho$: distribution function of $V_i$
- $D_i = 1_{\{\tau_i \leq t\}}, \ i = 1 \ldots n$ are conditionally independent given $V$
- \[ \frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} E[D_i \mid V] = P(\tau_i \leq t \mid V) = \bar{p} \]
Additive factor copula approaches

**Theorem**

For any fixed time horizon $t$, denote by $D_i = 1\{\tau_i \leq t\}$, $i = 1 \ldots n$ and $D_i^* = 1\{\tau_i^* \leq t\}$, $i = 1 \ldots n$ the default indicators corresponding to (resp.) $\rho$ and $\rho^*$, then:

$$\rho \leq \rho^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- This framework includes popular factor copula models:
  - One factor Gaussian copula - the industry standard for the pricing of CDO tranches
  - Double t: Hull and White(2004)
  - NIG, double NIG: Guegan and Houdain(2005), Kalemanova, Schmid and Werner(2007)
  - Double Variance Gamma: Moosbrucker(2006)
Archimedean copula


- $V$ is a positive random variable with Laplace transform $\varphi^{-1}$
- $U_1, \ldots, U_n$ are independent Uniform random variables independent of $V$
- $V_i = \varphi^{-1} \left( -\frac{\ln U_i}{V} \right), \ i = 1 \ldots n \ (\text{Marshall and Olkin (1988)})$
  - $(V_1, \ldots, V_n)$ follows a $\varphi$-archimedean copula
  - $P(V_1 \leq v_1, \ldots, V_n \leq v_n) = \varphi^{-1}(\varphi(v_1) + \ldots + \varphi(v_n))$
- $\tau_i = G^{-1}(V_i)$
  - $G$: distribution function of $\tau_i$
- $D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n$ independent knowing $V$
- $\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} E[D_i \mid V] = P(\tau_i \leq t \mid V)$
Archimedean copula

- Conditional default probability: $\tilde{p} = \exp \{-\varphi(G(t)V)\}$

<table>
<thead>
<tr>
<th>Copula</th>
<th>Generator $\varphi$</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clayton</td>
<td>$t^{-\theta} - 1$</td>
<td>$\theta \geq 0$</td>
</tr>
<tr>
<td>Gumbel</td>
<td>$(-\ln(t))^\theta$</td>
<td>$\theta \geq 1$</td>
</tr>
<tr>
<td>Franck</td>
<td>$-\ln \left[ (1 - e^{-\theta t})/(1 - e^{-\theta}) \right]$</td>
<td>$\theta \in \mathbb{R}^*$</td>
</tr>
</tbody>
</table>

Theorem

$\theta \leq \theta^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$
Archimedean copula

- Clayton copula
- Mixture distributions are ordered with respect to the convex order

\[
P(\tau_1 \leq t) = 0.08
\]
Hull, Predescu and White (2005)

- Consider \( n \) firms
- Let \( V_{i,t}, \ i = 1 \ldots n \) be their asset dynamics

\[
V_{i,t} = \rho V_t + \sqrt{1 - \rho^2} \tilde{V}_{i,t}, \ i = 1 \ldots n
\]

- \( V, \tilde{V}_i, \ i = 1 \ldots n \) are independent standard Wiener processes
- Default times as first passage times:

\[
\tau_i = \inf\{ t \in \mathbb{R}^+ | V_{i,t} \leq f(t) \}, \ i = 1 \ldots n, \ f : \mathbb{R} \to \mathbb{R} \text{ continuous}
\]

- \( D_i = 1_{\{\tau_i \leq T\}}, \ i = 1 \ldots n \) are conditionally independent given \( \sigma(V_t, \ t \in [0, T]) \)
For any fixed time horizon $T$, denote by $D_i = 1\{\tau_i \leq T\}$, $i = 1 \ldots n$ and $D_i^* = 1\{\tau_i^* \leq T\}$, $i = 1 \ldots n$ the default indicators corresponding to (resp.) $\rho$ and $\rho^*$, then:

$$\rho \leq \rho^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$
Distributions of Conditionnal Default Probabilities

- $\frac{1}{n} \sum_{i=1}^{n} D_i \rightarrow^{a.s.} \tilde{p}$
- $\frac{1}{n} \sum_{i=1}^{n} D_i^* \rightarrow^{a.s.} \tilde{p}^*$

Empirically, mixture probabilities are ordered with respect to the convex order: $
\tilde{p} \leq_{cx} \tilde{p}^*$

Portfolio size=$10000$
$X_0=0$
Threshold=$-2$
t=1 year
$\delta t=0.01$
$P(\tau_i \leq t)=0.033$

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Comparison results for credit risk portfolios
Multivariate Poisson model

- $\tilde{N}_t^i$: Poisson with parameter $\tilde{\lambda}$: idiosyncratic risk
- $N_t^i$: Poisson with parameter $\lambda$: systematic risk
- $(B_{ij})_{i,j}$ Bernoulli random variable with parameter $p$
- All sources of risk are independent
- $N_t^i = \tilde{N}_t^i + \sum_{j=1}^{N_t^i} B_{ij}, \ i = 1 \ldots n$
- $\tau_i = \inf\{t > 0|N_t^i > 0\}, \ i = 1 \ldots n$

Multivariate Poisson model

- Dependence structure of \((\tau_1, \ldots, \tau_n)\) is the Marshall-Olkin copula
- \(\tau_i \sim \text{Exp}(\tilde{\lambda} + p\lambda)\)
- \(D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n\) are conditionally independent given \(N_t\)
- \(\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} E[D_i \mid N_t] = P(\tau_i \leq t \mid N_t)\)
- Conditional default probability:
  \[
  \tilde{p} = 1 - (1 - p)^{N_t} \exp(-\tilde{\lambda}t)
  \]
Comparison of two multivariate Poisson models with parameter sets 
\((\tilde{\lambda}, \lambda, p)\) and \((\tilde{\lambda}^*, \lambda^*, p^*)\)

Supermodular order comparison requires equality of marginals:
\[
\tilde{\lambda} + p\lambda = \tilde{\lambda}^* + p^*\lambda^*
\]

3 comparison directions:
- \(p = p^*: \tilde{\lambda} \text{ v.s } \lambda\)
- \(\lambda = \lambda^*: \tilde{\lambda} \text{ v.s } p\)
- \(\tilde{\lambda} = \tilde{\lambda}^*: \lambda \text{ v.s } p\)
**Theorem \((p = p^*)\)**

Let parameter sets \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\) be such that \(\bar{\lambda} + p\lambda = \bar{\lambda}^* + p\lambda^*\), then:

\[ \lambda \leq \lambda^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_c \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*) \]

- Computation of \(E[(L_t - a)^+]\):
  - 30 names
  - \(M_i = 1, \ i = 1 \ldots n\)
- When \(\lambda\) increases, the aggregate loss increases with respect to stop-loss order
**Theorem** ($\lambda = \lambda^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda$, then:

$$p \leq p^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- Convex order for mixture probabilities

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**Multivariate Poisson model**
Multivariate Poisson model

Theorem \((\lambda = \lambda^*)\)

\[\text{Let parameter sets } (\bar{\lambda}, \lambda, p) \text{ and } (\bar{\lambda}^*, \lambda^*, p^*) \text{ be such that } \bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda, \text{ then:} \]

\[p \leq p^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \bar{p} \leq_{cx} \bar{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)\]

- Computation of \(E[(L_t - K)^+]\):
  - 30 names
  - \(M_i = 1, \ i = 1 \ldots n\)
- When \(p\) increases, the aggregate loss increases with respect to stop-loss order
Multivariate Poisson model

**Theorem** \((\overline{\lambda} = \overline{\lambda}^*)\)

Let parameter sets \((\overline{\lambda}, \lambda, p)\) and \((\overline{\lambda}^*, \lambda^*, p^*)\) be such that \(p \lambda = p^* \lambda^*\), then:

\[
p \leq p^*, \quad \lambda \geq \lambda^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)
\]

- Computation of \(E[(L_t - K)^+]\):
  - 30 names
  - \(M_i = 1, \ i = 1 \ldots n\)
- When \(p\) increases, the aggregate loss increases with respect to stop-loss order
When considering an exchangeable vector of default indicators, the conditional independence assumption is not restrictive thanks to de Finetti’s theorem.

The mixture probability (the factor) can be viewed as the loss of an infinitely granular portfolio.

We completely characterize the supermodular order between exchangeable default indicator vectors in terms of the convex ordering of corresponding mixture probabilities.

We show that the mixture probability is the key input to study the impact of dependence on CDO tranche premiums.

Comparison analysis can be performed with the same method within a large number of popular CDO pricing models.