Comparison results for homogenous credit portfolios

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General background context

- $n$ defaultable firms or policies
- $\tau_1, \ldots, \tau_n$ default times or claim occurrences
- $(D_1, \ldots, D_n) = (1\{\tau_1 \leq t\}, \ldots, 1\{\tau_n \leq t\})$ default or claim indicators
- $M_i$ loss given default or claim amount
- Aggregate loss or total claim amount:

$$L_t = \sum_{i=1}^{n} M_i 1\{\tau_i \leq t\}$$

- Stop Loss order results for $L_t$?
- Ordering of convex risk measures on $L_t$?
Specify the dependence structure between $D_1, \ldots, D_n$ which leads to:
- an increase of stop loss premiums
- an increase of convex risk measures

Exchangeability of $D_1, \ldots, D_n$
- De Finetti Theorem leads to a factor representation
- Simplifies comparison analysis

Comparison of Exchangeable Bernoulli random vectors

Application to several models of default and insurance
- Measure the impact of parameters governing the dependence
- Comparing copula, structural, multivariate Poisson models
Contents

1. Comparison of Exchangeable Bernoulli random vectors
   - De Finetti Theorem and Stochastic Orders
   - Review of literature
   - Main result

2. Application to Insurance and credit risk management
   - Multivariate Poisson model
   - Structural model
   - Factor copula models
     - Archimedean copula
     - Double $t$ copula

3. Comparison of different models
Definition (Exchangeability)

A random vector \((\tau_1, \ldots, \tau_n)\) is exchangeable if its distribution function is invariant by permutation: \(\forall \sigma \in S_n\)

\[
(\tau_1, \ldots, \tau_n) \overset{d}{=} (\tau_{\sigma(1)}, \ldots, \tau_{\sigma(n)})
\]

- Homogeneity assumption: default dates are assumed to be exchangeable
- Same marginals
De Finetti Theorem and Factor representation

**Theorem (De Finetti)**

Suppose that \(D_1, \ldots, D_n, \ldots\) is an exchangeable sequence of Bernoulli random variables, then there is a mixture probability measure \(\nu\) such that \(\forall n, \ \forall (d_1, \ldots, d_n) \in \{0, 1\}^n:\)

\[
P(D_1 = d_1, \ldots, D_n = d_n) = \int_0^1 p^{\sum_i d_i} (1 - p)^{n - \sum_i d_i} d\nu(p)
\]

- Usual De Finetti involves infinite sequences
  - Finite exchangeability only leads to a sign measure Jaynes 1986
De Finetti Theorem and Factor representation

Denote by $F$ the distribution function of $\nu$: $F(p) = \nu([0, p])$

There exists a random factor $\tilde{p}$ distributed as $F$ such that:

- $D_1, \ldots, D_n$ are independent knowing $\tilde{p}$
- $\tilde{p}$ is a.s unique and such that:

\[
\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s} \tilde{p} \quad \text{as} \quad n \to \infty
\]
Stochastic orders

- $X \leq_{cx} Y$ if $E[f(X)] \leq E[f(Y)]$ for all convex functions $f$
- $X \leq_{icx} Y$ if $E[f(X)] \leq E[f(Y)]$ for all increasing convex functions $f$
- $X \leq_{sl} Y$ if $E[(X - K)^+] \leq E[(Y - K)^+]$ for all $K \in \mathbb{R}$
  - stop loss order and icx-order are equivalent
  - $X \leq_{sl} Y$ and $E[X] = E[Y] \iff X \leq_{cx} Y$
- $X \leq_{less-dangerous} Y$ if there exists $x_0$ such that $F_X(x) \leq F_Y(x)$ for all $x \leq x_0$ and $F_X(x) \geq F_Y(x)$ for all $x \geq x_0$ and moreover $E[X] \leq E[Y]$
  - less dangerous order $\Rightarrow$ icx-order or stop loss order
Stochastic orders

Definition (Supermodular function)

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is supermodular if for all $x, y \in \mathbb{R}^n$

$$f(x \land y) + f(x \lor y) \geq f(x) + f(y) \quad \text{when}$$

$x \land y = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$ and $x \lor y = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$

- $X \leq_{sm} Y$ if $E[f(X)] \leq E[f(Y)]$ for all supermodular functions $f$
Review of literature

Shaked and Shanthikumar (1994)
Stochastic Orders and Their Applications.

Müller and Stoyan (2002)
Comparison Methods for Stochastic Models and Risks.

Denuit, Dhaene, Goovaerts and Kaas (2005)
Actuarial Theory for Dependent Risks - Measures, Orders and Models.
Review of literature

Müller (1997)
Stop-loss order for portfolios of dependent risks.

\[(X_1, \ldots, X_n) \leq_{sm} (Y_1, \ldots, Y_n) \Rightarrow \sum_{i=1}^{n} M_i X_i \leq_{sl} \sum_{i=1}^{n} M_i Y_i\]

Bäuerle and Müller (2005)
Stochastic orders ans risk measures: Consistency and bounds

\[X \leq_{icx} Y \Rightarrow \rho(X) \leq \rho(Y)\]

for all law-invariant, convex risk measures \(\rho\)

Lefèvre and Utev (1996)
Comparing sums of exchangeable bernoulli random variables.

\[\tilde{p} \leq_{icx} \tilde{p}^* \Rightarrow \sum_{i=1}^{n} D_i \leq_{sl} \sum_{i=1}^{n} D_i^*\]
Supermodular order for Exchangeable Bernoulli random vectors

Theorem

Let $\mathbf{D} = (D_1, \ldots, D_n)$ and $\mathbf{D}^* = (D_1^*, \ldots, D_n^*)$ be two exchangeable Bernoulli random vectors with (resp.) $F$ and $F^*$ as mixture distributions. Then:

$$F \leq_{cx} F^* \Rightarrow \mathbf{D} \leq_{sm} \mathbf{D}^* \quad \text{and}$$
$$F \leq_{icx} F^* \Rightarrow \mathbf{D} \leq_{ism} \mathbf{D}^*$$
Supermodular order for Exchangeable Bernoulli random vectors

Theorem

Let $D_1, \ldots, D_n, \ldots$ and $D_1^*, \ldots, D_n^*, \ldots$ be two exchangeable sequences of Bernoulli random variables. We denote by $F$ (resp. $F^*$) the distribution function associated with the mixing measure. Then,

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*), \forall n \in \mathbb{N} \Rightarrow F \leq_{cx} F^*. \quad (1)$$
Ordering of CDO tranche premiums

- CDO: Collateralized Debt Obligation
  - insurance contract which covers portfolio losses $L_t$
  - in a certain tranche $[\alpha, \beta]$ of the total notional

- Tranche premiums only involves call options on the accumulated losses $L_t$:
  $$E[(L_t - \alpha)^+] - E[(L_t - \beta)^+]$$
Ordering of CDO tranche premiums

Burtschell, Gregory, and Laurent (2005a)  
A Comparative Analysis of CDO Pricing Models
- Supermodular order for some factor copula models
  - Gaussian copula
  - Student $t$ copula
  - Clayton copula
  - Marshall-Olkin copula

Burtschell, Gregory, and Laurent (2005b)  
Beyond the Gaussian Copula: Stochastic and Local Correlation
- Stochastic correlation
Multivariate Poisson model

- $\tilde{N}_t^i$ Poisson with parameter $\bar{\lambda}$: idiosyncratic risk
- $N_t$ Poisson with parameter $\lambda$: systematic risk
- $(B^i_j)_{i,j}$ Bernoulli random variable with parameter $p$
- All sources of risk are independent
- $N_t^i = \tilde{N}_t^i + \sum_{j=1}^{N_t} B^i_j$, $i = 1 \ldots n$
- $\tau_i = \inf \{ t > 0 | N_t^i > 0 \}$, $i = 1 \ldots n$

<table>
<thead>
<tr>
<th>$\tilde{N}_t^i = 0$</th>
<th>$\tilde{N}_t^i = 0$</th>
<th>$\tilde{N}_t^i = \tilde{N}_t^1 = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_t = 1$, $B_{i,1} = 0$</td>
<td>$N_t = 2$, $B_{i,2} = 0$</td>
<td>$N_t = N_2 = B_{t,2} = B_{t,3} = B_{t,\emptyset} = 1$</td>
</tr>
</tbody>
</table>
Multivariate Poisson model

- \( \tau_i \sim \text{Exp}(\bar{\lambda} + p\lambda) \)
- Dependence structure of \((\tau_1, \ldots, \tau_n)\) is the Marshall-Olkin copula
- \( D_i = 1_{\{\tau_i \leq t\}}, \ i = 1 \ldots n \) are independent knowing \( N_t \)
- \( \frac{1}{n} \sum_{i=1}^{n} D_i \overset{a.s.}{\to} E[D_i \mid N_t] = P(\tau_i \leq t \mid N_t) \)
- Conditional default probability:
  \[
  \tilde{p} = 1 - (1 - p)^{N_t} \exp(-\bar{\lambda}t)
  \]
Comparison of two multivariate Poisson models with parameter sets 
\((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\)

Supermodular order comparison requires equality of marginals:
\[\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda^*\]

3 comparison directions:
- \(p = p^*:\ \bar{\lambda} \text{ v.s } \lambda\)
- \(\lambda = \lambda^*:\ \bar{\lambda} \text{ v.s } p\)
- \(\bar{\lambda} = \bar{\lambda}^*:\ \lambda \text{ v.s } p\)
Theorem \((p = p^*)\)

Let parameter sets \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\) be such that 
\[
\bar{\lambda} + p\lambda = \bar{\lambda}^* + p\lambda^*,
\]
then:
\[
\lambda \leq \lambda^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq c_x \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)
\]

- Computation of \(E[(L_t - K)^+]\):
  - 30 names
  - \(M_i = 1, \ i = 1 \ldots n\)
- Stop-loss premiums are ordered...
Theorem ($\lambda = \lambda^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that

$$\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda,$$

then:

$$p \leq p^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \bar{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- Less Dangerous order for mixture distributions
Let parameter sets $(\lambda, \lambda, p)$ and $(\lambda^*, \lambda^*, p^*)$ be such that $\lambda + p\lambda = \lambda^* + p^*\lambda$, then:

$$p \leq p^*, \quad \lambda \geq \lambda^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- Computation of $E[(L_t - K)^+]$:
  - 30 names
  - $M_i = 1, \; i = 1 \ldots n$
- Stop-loss premiums are ordered...
Multivariate Poisson model

Theorem ($\bar{\lambda} = \bar{\lambda}^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $p\lambda = p^*\lambda^*$, then:

$$p \leq p^*, \lambda \geq \lambda^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- Computation of $E[(L_t - K)^+]$:
  - 30 names
  - $M_i = 1, \ i = 1 \ldots n$
- Stop-loss premiums are ordered...
Hull, Predescu and White (2005)

- Consider $n$ firms
- Let $X^i_t$, $i = 1\ldots n$ be their asset dynamics
  \[ X^i_t = \sqrt{\rho} W_t + \sqrt{1 - \rho} W^i_t, \quad i = 1\ldots n \]
- $W$, $W^i$, $i = 1\ldots n$ are independent standard Wiener processes
- Default times as first passage times:
  \[ \tau_i = \inf \{ t \in \mathbb{R}^+ | X^i_t \leq f(t) \}, \quad i = 1\ldots n, \quad f : \mathbb{R} \rightarrow \mathbb{R} \text{ continuous} \]
- $D_i = 1_{\{\tau_i \leq T\}}, i = 1\ldots n$ are independent knowing $\sigma(W_t, t \in [0, T])$
Comparison of Exchangeable Bernoulli random vectors
Application to Insurance and credit risk management
Comparison of different models
Conclusion

Multivariate Poisson model
Structural model
Factor copula models

Structural Model

Theorem

For any fixed time horizon $T$, denote by $D_i = 1_{\{\tau_i \leq T\}}$, $i = 1 \ldots n$ and $D_i^* = 1_{\{\tau_i^* \leq T\}}$, $i = 1 \ldots n$ the default indicators corresponding to (resp.) $\rho$ and $\rho^*$, then:

$$\rho \leq \rho^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$
Comparison results for homogenous credit portfolios

- \( \frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} \tilde{p} \)
- \( \frac{1}{n} \sum_{i=1}^{n} D_i^* \xrightarrow{a.s.} \tilde{p}^* \)
- Less Dangerous order for mixture distributions
Archimedean copula

Cossette, Gaillardetz, Marceau and Rioux(2002), Wei and Hu(2002)

- $V$ is a positive random variable with Laplace transform $\varphi^{-1}$
- $U_1, \ldots, U_n$ are independent Uniform random variables independent of $V$
- $V_i = \varphi^{-1} \left( -\ln \frac{U_i}{V} \right)$, $i = 1 \ldots n$
  - $(V_1, \ldots, V_n)$ follows a $\varphi$-archimedean copula
  - $P(V_1 \leq v_1, \ldots, V_n \leq v_n) = \varphi^{-1}(\varphi(v_1) + \ldots + \varphi(v_n))$
- $\tau_i = G^{-1}(V_i)$
  - $G$: distribution function of $\tau_i$
- $D_i = 1\{\tau_i \leq t\}$, $i = 1 \ldots n$ independent knowing $V$
- $\frac{1}{n} \sum_{i=1}^n D_i \xrightarrow{a.s.} E[D_i \mid V] = P(\tau_i \leq t \mid V)$
- Conditional default probability:
  $$\tilde{p} = \exp \{-\varphi(G(t)V)\}$$
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Archimedean copula

<table>
<thead>
<tr>
<th>Copula name</th>
<th>Generator $\varphi$</th>
<th>$V$-distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clayton</td>
<td>$t^{-\theta} - 1$</td>
<td>Gamma($1/\theta$)</td>
</tr>
<tr>
<td>Gumbel</td>
<td>$(-\ln(t))^{\theta}$</td>
<td>$\alpha$-Stable, $\alpha = 1/\theta$</td>
</tr>
<tr>
<td>Franck</td>
<td>$-\ln \left[(1 - e^{-\theta t})/(1 - e^{-\theta})\right]$</td>
<td>Logarithmic series</td>
</tr>
</tbody>
</table>

Theorem

$$\theta \leq \theta^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

The proof derived from the following result:

Theorem

Let $V$ and $V^*$ be two positive random variables. Denote by $\varphi^{-1}$ et $\psi^{-1}$ their Laplace transform. Consider $\tilde{p} = \exp(-\varphi(G(t))V)$ and $\tilde{p}^* = \exp(-\psi(G(t))V^*)$, the corresponding conditional default probabilities, then:

$$\varphi \circ \psi^{-1} \in \mathcal{L}_\infty^* = \{ f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ | (-1)^{n-1}f^{(n)} \geq 0 \ \forall \ n \geq 1 \} \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^*$$
Archimedean copula

- Clayton copula
- Less Dangerous order for mixture distributions

$\theta \in \{0.01; 0.1; 0.2; 0.4\}$

$P(\tau_i \leq t) = 0.08$
Archimedean copula

- Previous result is consistent with
  - Cossette, Gaillardetz, Marceau and Rioux (2002)
- Computation of $E[(L_t - u)^+]$ with $L_t = \sum_{i=1}^{20} M_i D_i$
- $(D_1, \ldots, D_{20})$ follows a Clayton copula with parameter $\alpha$
- $P(D_i = 1) = 0.05$
- $M_i \sim \text{Gamma}(1, 2)$

![Graph showing stop-loss premiums for different $\alpha$ values](attachment:image.png)
Double $t$ copula

- $V \sim t(\nu)$, $\tilde{V}_i \sim t(\bar{\nu})$ Student with $\nu$ (resp.) $\bar{\nu}$ degree of freedom

\[ V_i = \rho \left( \frac{\nu - 2}{\nu} \right)^{1/2} V + \sqrt{1 - \rho^2} \left( \frac{\bar{\nu} - 2}{\bar{\nu}} \right)^{1/2} \tilde{V}_i \]

- $\tau_i = G^{-1}(H_\rho(V_i))$, $i = 1 \ldots n$
  - $G$: distribution function of $\tau_i$
  - $H_\rho$: distribution function of $V_i$

- $D_i = 1_{\{\tau_i \leq t\}}$, $i = 1 \ldots n$ independent knowing $V$

\[ \frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} \mathbb{E}[D_i \mid V] = P(\tau_i \leq t \mid V) \]

- Conditional default probability:

\[ \tilde{p} = t_{\bar{\nu}} \left( \frac{\bar{\nu}}{\bar{\nu} - 2} \right)^{1/2} \frac{H_\rho^{-1}(G(t)) - \rho \left( \frac{\nu - 2}{\nu} \right)^{1/2} V}{\sqrt{1 - \rho^2}} \]
Double $t$ copula

Theorem

$$\rho \leq \rho^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$
Comparison of different models

- Stop loss premium: \( E[(L_t - K)^+] \) with \( L_t = \frac{1}{n} \sum_{i=1}^{n} D_i \)
- Comparison criteria:
  - same default marginals for all models
  - dependence parameters set to get equal premiums for \( K = 0.03 \)

### Graphs

- **Left Graph**: Comparison of stop loss premiums for different models with 30 names, t=5 years, \( P(\tau \leq t) = 0.08 \).
  - Red stars: Double \( \text{NG}(1) - \text{NG}(1) \), \( \rho = 0.54 \)
  - Red line: Double \( t(4) - t(4) \), \( \rho = 0.54 \)
  - Green line: Clayton, \( \theta = 0.19 \)
  - Blue line: Gaussian, \( \rho = 0.54 \)
  - Pink line: Mult-Poisson, \( \lambda = 0.06 \), \( \rho = 0.1 \), \( \lambda = 0.11 \)

- **Right Graph**:
  - Red line: Double \( t(4) - t(4) \), \( \rho = 0.54 \)
  - Blue line: Gaussian, \( \rho = 0.54 \)
Comparison of Exchangeable Bernoulli random vectors
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Comparison of different models

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Comparison results for homogenous credit portfolios
Conclusion

- Characterization of supermodular order for exchangeable Bernoulli random vectors
- Comparison of CDO tranche premiums or reinsurance premiums in the individual life model
- Unified way of presenting default risk models...