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A Swap Curve for insurance Risk Management, based on no arbitrage short-rates models

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Abstract We introduce a model for the yield curve, whose static discount factors rely on the closed-form formulas for zero-coupons available in exogenous (also known as no arbitrage) short rate models. After their calibration, the spot rates can be extrapolated to unobserved maturities by converging to a fixed ultimate forward rate. Such an extrapolation of the yield curve is useful for discounting long-term liabilities cash-flows. If one is interested in no arbitrage pricing, then she can use simulations under the risk neutral probability of the corresponding exogenous short rates model. Otherwise, yield curve forecasts can be obtained under the historical probability, by applying a Functional Principal Components Analysis to the model's parameters.

Keywords Discount curve · Insurance · Extrapolation · Forecasting

1 Context

The new Solvency II directive defines the calculation of European insurers' technical provisions as the sum of two components, the Best Estimate Liabilities (BEL) and the Risk Margin (RM). The Best Estimate Liabilities (BEL) are defined as the average discounted value of the insurers future cash-flows, weighted by their probability of occurrence. The Risk Margin is a supplemental amount required for covering the non-hedgeable risks, by involving a capital lockup.

In order to discount the cash-flows relevant in the calculation of the BEL and Risk Margin, an *appropriate* term structure of discount factors is needed. From the no arbitrage pricing theory developed by [13], and currently widely used in insurance *market consistent* pricing of liabilities, the zero rates related

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to the stochastic discount factors have to be *risk-free*. That is, free from any counterparty credit risk.

There is no easy answer to the question of defining such a *risk-free* rate for insurance liabilities. It could be related the insurers own assets return, where the liabilities are *perfectly* backed by the assets. But in a *market consistent* approach as required in Solvency II, since not every liability is perfectly backed by the assets, a more fundamental *risk-free* rate also needs to be derived. Deriving such a *common risk-free* rate from market-quoted instruments is also aimed at increasing transparency and comparability of balance sheets across European countries.

For years, in banking, the construction of a term structure of *risk-free* discount factors was based on the assumption that banks are not subject to counterparty credit risk when lending to each other, and liquidity was not an issue. In this context, interbank rates (loosely called LIBOR hereafter) were seen as the best proxies for *risk-free* rates.

From the 2007-2008 financial crisis onwards, the spreads between swaps rates with different tenors started to widen, partly due to the increased reticence of banks to lend to each other. Today, LIBOR is no longer considered as a proxy for *risk-free* rates, and market operators have increasingly started to use Overnight Interest Swaps (OIS) discounting (see [16] for example).

Comparatively in the European Insurance market, throughout the quantitative impact studies (the QIS) leading to Solvency II, the questions of *risk-free* term structure construction for valuation have been tackled for years by the CEIOPS and later by the EIOPA (see [6] for example). The difficulty in defining a fundamental *risk-free* rate for the insurance market, mainly arises from the fact that a pure market *risk-free* rate could introduce a lot of unwanted market volatility into the insurers balance sheet. Hence, this discount curve has been adjusted with different spreads through the QIS, and until its most recent specification, making it somewhat, less *consistent* with the market.

As of June 2015 (see [11]), the term structure of discount factors for insurers liabilities cash-flows is indeed derived from LIBOR EUR swap (IRS hereafter) rates, as the market for vanilla swaps is considered as Deep, Liquid, and Transparent (the DLT assumption). A credit risk adjustment (CRA) is prescribed by the directive, consisting in a parallel shift applied to LIBOR swap rates. The parallel shift shall not be lower than -35bps or greater than -10 bps. Furthermore, a matching adjustment and a volatility adjustment are other optional parallel shifts which could be applied to the constructed curve.

The volatility adjustment is designed to be used in case of a crisis, causing the widening of sovereign or corporate bonds spreads. On the other hand, the matching adjustment is used in cases where the liabilities are predictable, that is, almost *perfectly* backed. In this paper, we focus on discount curve construction. The matching premium and the volatility adjustment are not further discussed.

Beyond the data and curve adjustments concerns, and considering curve construction methods, [1] distinguish between two types of methods: *best fit* methods, and *exact fit* methods. Best fit methods, such as [20] and [25] are widely used by central banks. Exact fit methods such as cubic splines methods on the other hand, generally have at least as much parameters as input market products, and provide an exact fit to market data.

While the latter type of methods would be adapted for no arbitrage pricing and trading, the former type are useful for forecasting the yield curve in real world probability (see [10] for example). They fit the curve parsimoniously with a few parameters; in an attempt to mimic the factors explaining the variance of the yield curve changes (see [19] for details). There is another class of models, which combine the idea of using a factors structure, which the absence of dynamic arbitrages in the curve diffusion, see [7] for example.

The extrapolation of the constructed curve is also an important subject matter for insurers and pension funds. Indeed, some of their liabilities cash-flows may have very long maturities, spanning beyond the longest liquid maturities available for market-quoted instruments. The question is, how would spot rates for such long maturities be determined?

Currently in Solvency II, the construction and extrapolation of the swap curve is made by using the Smith-Wilson method described in [24] and in the technical specifications [11]. The Smith-Wilson method constructs the swap curve by exactly fitting the market IRS rates adjusted from a CRA. After a chosen maturity - the last liquid point (LLP), currently 20 years -, the forward rate is forced by regulatory rules, to converge at an exogenously specified speed to a fixed long term level called the Ultimate Forward Rate (UFR). The UFR is currently derived as the sum of expected Euro inflation and expected real rates. It is currently equal to 4.2%.

For discount curve construction and extrapolation, we propose a method which relies on closed-form formulas for discount factors available in exogenous (or no arbitrage) short rates model. It could be both an *exact fit* and *best fit* method, depending on the data at hand, and on how the curve is calibrated to these data. In this framework, the time-varying function ensuring an exact fit to market implied discount factors in exogenous short rates models is considered to be a piecewise constant function, whose steps become model's parameters. The interpolation of the curve at dates comprised between quoted maturities directly comes from the properties of the model. Pseudo-discount curves can also be constructed in a dual curve environment, by using our method, along with the recipes described for example in [26] and [1].

After the static discount curve calibration to market data, the curve can be extrapolated to unobserved maturities by converging to an *ultimate forward rate*. Extrapolation is done by using the same model that the one used for interpolation. On this particular point, our model is hence closer to the [24] model than to models which use different methods for interpolation and extrapolation (such as cubic splines for interpolation, and a modified version of [20] for extrapolation). We describe ways to either derive an UFR from the data, or to constraint the model to converge to a given UFR.

When it comes to forecasting and/or simulation, if one is interested in no arbitrage pricing, then she can use simulations under a risk neutral probability of the corresponding, consistent (in the sense of [4]) exogenous short rates model. Otherwise, forecasts of the yield curve under the historical probability can be obtained by making use of a functional principal components analysis.

In the next section, we describe the model proposed for discount curve construction and extrapolation, and explain how it could be calibrated to market data. Then, we explain how to obtain forecast of the discount curve, by using the model's parameters. To finish, some numerical examples based on [12], [2], [3], [1] are presented.

2 Curve construction and extrapolation

The model proposed for discount curve construction and extrapolation relies on exogenous short rates models, also known as no-arbitrage short rates models. These models incorporate a time-varying function, constructed in such a way that, the model's implied discount factors match exactly the market implied discount factors.

In general, the market implied discount factors are derived by using a static (*exact fit*) curve calibration method, such as cubic splines or [24]. In our framework, if one is interested in no arbitrage pricing, the curve calibration method is consistent with the corresponding diffusion model, in the sense of [4].

As an illustration, the exogenous short rates model that we consider here under a risk-neutral probability measure $\mathbb{Q}$ is an Hull-White extended Vasicek ([14] and [15]):

$$dX_t = a(b(t) - X_t)dt + \sigma dW_t \tag{1}$$

where $(W_t)_{t \geq 0}$ is a standard brownian motion under a risk-neutral probability $\mathbb{Q}$. But the methodology can be directly extended to any other exogenous short rates model with closed form formulas for discount factors, including multi-factor gaussian short rate models[1].

a controls the speed of mean reversion of the short rates, and σ is their volatility. $t \mapsto b(t)$ is a time-varying deterministic function, used for fitting exactly the current market discount factors $P^M(0,t)$, and controlling the convergence of long-term rates. Typically in order to obtain an *exact* fit in the Hull-White extended Vasicek model from equation (1), we have to choose:

$$b(t) = \frac{1}{a}\frac{df^M}{dt}(0,t) - f^M(0,t) + \frac{1}{2}\frac{\sigma^2}{a^3}\left(1 - e^{-2at}\right) \tag{2}$$

[1] Even though in this case, there would be more free parameters

where $t \mapsto f^M(0,t)$ are the market implied instantaneous forward rates.

In the absence of arbitrage opportunities, the value at time t_0 of a discount factor with maturity t is given by

$$P(t_0,t) = \mathbb{E}_{\mathbb{Q}}\left[exp\left(-\int_{t_0}^{t} X_u du\right)|\mathcal{F}_{t_0}\right] \tag{3}$$

where $\mathcal{F}$ is the natural filtration of the short-rate process X.

For the model in equation (1), and by using (3), closed-form formulas are available for discount factors (and implicitly for discount rates and instantaneous forward rates). Indeed, assuming that $X_{t_0} = X_0$, we can write:

$$P(t_0,t) = \exp\left(-X_0\phi(t-t_0) - a\int_{t_0}^{t} b(u)\phi(t-u)du - c\psi(t-t_0)\right) \tag{4}$$

where ϕ and ψ are defined as

$$\phi(s) := \frac{1}{a}\left(1 - e^{-as}\right), \tag{5}$$

$$\psi(s) := -\int_0^s \left(\frac{\sigma^2}{2}\phi^2(s-\theta)\right) d\theta. \tag{6}$$

We derive $P^M(0,t)$ from market inputs, based on (4), (5), (6). By doing so, we avoid the use of cubic splines methods to obtain $P^M(0,t)$, and create a consistent discount curve construction, in the sense of [4].

Deriving $P^M(0,t)$ in this framework assumes the time-varying function $t \mapsto b(t)$ to be a piece-wise constant function, whose steps are derived from vanilla (IRS) or overnight swaps (OIS) cash-flows. A similar idea was applied by [23] for example, to the model from [9]. We let $T_1, \ldots, T_n$, be the maturities of market quoted IRS, with Credit Risk Adjustment (CRA), or OIS. Considering that the function $t \mapsto b(t)$ is piecewise-constant, with:

$$b(t) = b_i, \;\; \text{for } T_{i-1} \leq t < T_i, \;\; i = 1, \ldots, n \tag{7}$$

$$b(t) = b_{n+1}, \;\; \text{for } t \geq T_n \tag{8}$$

and $T_0 = t_0 = 0$, we are able to derive closed-form formulas for the discount factors, taking into account the discrete b_i's, swap maturities, and using (4):

$$P^M(0,t) = exp\left(-X_0\phi(t) - I_{n+1}(t) + \frac{\sigma^2}{2a^2}\left(t - \phi(t)\right) - \frac{\sigma^2}{4a}\phi^2(t)\right) \tag{9}$$

where:

$$I_{n+1}(t) = \sum_{k=1}^{n} b_k \left(\xi(t - T_{k-1} \wedge t) - \xi(t - T_k \wedge t)\right) \tag{10}$$

for any $t \leq T_n$, and for $t > T_n$:

$$I_{n+1}(t) = \sum_{k=1}^{n} b_k \left(\xi(t - T_{k-1} \wedge t) - \xi(t - T_k \wedge t)\right) + b_{n+1}\xi(t - T_n \wedge t) \quad (11)$$

and

$$\xi(s) = s - \phi(s), \; s \geq 0 \quad (12)$$

Proof: Using the fact that in this framework $t \mapsto b(t)$ is piecewise constant, the integral $\int_{t_0}^{t} b(u)\phi(t-u)du$ in equation (4) can be written as

$$\sum_{k=1}^{n+1} \int_{T_{k-1}\wedge t}^{T_k \wedge t} b(u)\phi(t-u)du = \sum_{k=1}^{n+1} b_k \int_{T_{k-1}\wedge t}^{T_k \wedge t} \phi(t-u)du \;\square$$

One could notice that in this framework, the interpolation of the curve at intermediary dates between quoted swaps maturities is not necessary, and directly comes from the properties of the model (giving $P^M(t_0, t)$ for all t in equations (9) and (11)). Plus, as suggested by (2) and considering that the forward curve is generally assumed to be well-behaved, we do not generally expect the b_i's to be highly volatile.

In the next section, we explain how to calibrate the model's parameters in different situations, for the construction of OIS and IRS (with CRA) discount curves. Pseudo-discount curves could also be constructed in a dual curve environment, by using this method along with the recipes described for example in [26] and [1]. Dual curve construction is not described in this paper; the interested reader can refer to [26] and [1] for further details.

2.1 Calibration of the liquid part

Considering that there are N quoted swaps used for constructing the discount curve as of today, and at most M coupon payment dates for all the swaps, we let V be the vector of current values for the market swaps with length equal to N. C is the $N \times M$ matrix containing in each row, the swaps' coupon payments. And P the vector of discount factors that we are trying to derive, having a length equal to M.

Three methods might be envisaged for calibrating the model, depending on the data at hand:

- A method to be used **if an exact fit is required**, and can be found. That is, if we require $V = CP$
- A method to be used **if an exact fit cannot necessarily be found**, but approximated

– A method to be used **when the dataset is noisy, and a smooth curve is required**

These 3 methods are described hereafter, and numerical examples can be found in section 4.

– **If an exact fit is required**, then it is possible to guess *reasonable* values for a and σ (say, a between 0.05 and 1, and σ between 1% and 5%), and use an iterative curve calibration (also known as *bootstrapping*, but different from statistical bootstrap resampling) to solve $V = CP$.

  This type of method was used for any vanilla swap before the 2007 crisis, no matter its tenor. It is currently relevant only for extracting discount factors from OIS which are considered to be perfectly collateralized, or for Solvency II. Single curve construction in Solvency II, is currently applied to IRS, along with a parallel CRA, comprised between 10bps and 35bps.

  In order to describe the curve's calibration procedure, we will use a formulation similar to the one in [3]. We let $T_1 < \ldots < T_n$ be the maturity dates of OIS or IRS minus CRA, with the same currency on both legs. The swap paiment dates are

$$t_j = j \times x$$

  where $x \in \{1\,month, 3\,month, 6\,month, 1\,year\}$ is the swap's tenor, and $j \in \{1, \ldots, n/x\}$.

  The single curve construction, in the specific Hull & White-consistent case treated in this paper, is made as follows:

  1. Guess a and σ: any *reasonable* values for a and σ will produce an *exact* fit for discount factors and discount rates
  2. *Loop on i:* At each step T_i corresponding to the i^{th} input swap maturity, suppose that the discount factors and b_js are known for any $t_j < T_i$
  3. Make a guess for b_i
  4. Use formula (9), to derive the discount factors at intermediate swap paiment dates: $T_{i-1} \leq t_j \leq T_i$. No interpolation is required.
  5. Calculate V_i, the value of the i^{th} swap. While $V_i \neq 0$ return to point 2. Typically, the points 3 to 5 are solved iteratively with a root search algorithm.

     Another way for picking a and σ might be to calibrate the short rates model to a set of market prices of caps and swaptions.

– **If no solution is available for equation** $V = CP$ by iterative curve calibration, then similarly to [2], it is possible to search for P minimizing:

$$\frac{1}{2N}\left(V - CP\right)^T W^2 \left(V - CP\right) \tag{13}$$

W, a diagonal matrix of weights is used. These weights are based on inverse duration, such as proposed by [5]; with elements:

$$w_j = \frac{1/d_j}{\sum_{j=1}^{N} 1/d_j} \tag{14}$$

Weights such as w_j's are commonly used to give more importance to the short end of the curve, which is hence fitted more accurately. But other weighting schemes might be envisaged.

– **If the swaps data are noisy, or if one is interested in fitting smoothly noisy bonds data** a third method could be envisaged. It consists in penalizing the possibly large changes in forward rates' (approximate) second derivatives and/or in b_is. The objective function to be minimized is:

$$\frac{1}{2N}(V - CP)^T W^2 (V - CP) + \lambda_1 \sum_{i=1}^{N} (f_i^{''} - f_{i-1}^{''})^2 + \lambda_2 \sum_{i=1}^{N} (b_i - b_{i-1})^2 \tag{15}$$

where $f_i^{''}$ is the approximate second derivative (using finite differences) of the discrete forward rates at time T_i. Typically, λ_1 and λ_2 can be found by cross-validation. An example can be found in section 4.

2.2 Curve extrapolation

Using the Hull and White extended Vasicek model, it is possible to derive the instantaneous forward rates from the discount factors formula. We can write:

$$f(0,t) = -\frac{\partial log(P(0,t)}{\partial t} = X_0 e^{-at} + a \int_0^t e^{-a(t-u)} b(u) du - \frac{\sigma^2}{2} \phi^2(t) \tag{16}$$

Hence, in our framework, using the fact that $t \mapsto b(t)$ is piecewise constant, we can also write:

$$f^M(0,t) = X_0 e^{-at} + a \sum_{i=1}^{n} b_i \left[\phi(t - T_{i-1} \wedge t) - \phi(t - T_i \wedge t)\right] + a b_{n+1} \phi(t - T_n \wedge t) - \frac{\sigma^2}{2} \phi^2(t) \tag{17}$$

This formula directly provides an input for the simulation of Hull & White short rates, with parameters a, σ and $b_1, \ldots, b_n$ previously calibrated to market data.

Hence, letting t grow to ∞, we have:

$$f^M(0,\infty) = b_{n+1} - \frac{\sigma^2}{2a^2} \tag{18}$$

And if we assume that the UFR is exogenously chosen, and denote it by f_∞, we are able to derive the parameter b_{n+1} as:

$$b_{n+1} = f_\infty + \frac{\sigma^2}{2a^2} \tag{19}$$

This enables to re-write equation (11), when extrapolation is required, as:

$$I_{n+1}(t) = \sum_{k=1}^{n} b_k \left(\xi(t - T_{k-1} \wedge t) - \xi(t - T_k \wedge t) \right) + \left(f_\infty + \frac{\sigma^2}{2a^2} \right) \xi(t - T_n \wedge t) \tag{20}$$

If **a fixed *ultimate forward rate* (UFR) is defined exogenously**, one can increase or decrease the parameter a, to achieve a convergence of $f^M(0,t)$ to f_∞ at a pre-specified maturity. A period of convergence τ_{cv} after the *Last Liquid Point* (LLP) is defined. Starting from a low value such as $a = 0.1$, a is increased until:

$$f^M(0, LLP + \tau_{cv}) = f_\infty$$

or

$$|f^M(0, LLP + \tau_{cv}) - f_\infty| < tol$$

for a given σ, and a given numerical tolerance tol.

Otherwise, an ***ultimate forward rate* (UFR) can be derived from market data**. A static discount curve is fitted to a fraction of the quoted swaps available, called the *training* set. After the construction of the curve on this fraction of the data, we evaluate how well, when extrapolated to a given exogenous UFR, it would price the remaining swaps in a *test* set.

The LLP provided by the prudential authority (currently, a maturity 20 years), could be used to define the frontier between the *training* and *test* set. Otherwise, one can define a percentage of the swaps data to be used as a *training* dataset, for example 80% or 90% of the available swaps.

Both of these methods for curve extrapolation are applied in the numerical examples, in section 4.

3 Short term forecasting with Functional PCA

The idea that a few principal components explain a major part of the changes in bonds returns originates from [19]. This idea is now well accepted and applied to yield curve forecasting; the interested reader could refer to [10] or [7] for example.

We use a similar rationale, but apply it somewhat differently. The changes in the swap curve over time, are explained by the changes observed in the calibrated parameters b_is over time. Considering the fact that our model for fitting each cross section of yields is already *overparametrized* (as it uses at least as much parameters as swap rates available in the input dataset), the use of models such as an unrestricted Vector Autoregressive (VAR) to predict the b_is could lead to poor forecasts, with high variance.

Functional Principal Components Analysis in the spirit of [22] and [21], and more precisely Functional Principal Components Regression, was hence seen as one of the most immediate candidate to achieve a reduction of the problem's dimension. This method is already used for example by [18] for forecasting log mortality rates.

We consider functional data of the form:

$$b_x^{a,\sigma}(t) \tag{21}$$

These are the parameters obtained by fitting each cross section of swap rates; observed at increasing times $t \in \{t_1, \ldots, t_N\}$, for increasing maturities $x \in \{x_1, \ldots, x_p\}$. The calibration method is the one described in section 2.1, with a and σ kept fixed over time.

Finding the Functional Principal Components

Using the approach described in [21], we let $\mathbf{B}$ be the matrix containing at line i and column j:

$$\mathbf{B}_{i,j} = b_{x_j}^{a,\sigma}(t_i) \tag{22}$$

With $i = 1, \ldots, N$ and $j = 1, \ldots, n$, $n > p$. For each cross section of b_is observed at time t_i, a cubic spline interpolation has been applied to $x \mapsto b_x^{a,\sigma}(t_i)$, so that the b_is values are now equally spaced on a larger grid spanning $[x_1, x_p]$. Letting w be the fixed interpolation step applied to $x \mapsto b_x^{a,\sigma}(t_i)$ on $[x_1, x_p]$, and:

$$\mathbf{V} = \frac{1}{N}\mathbf{B'B} \tag{23}$$

We are looking for the vectors $\xi^{a,\sigma}$, the (approximate) functional principal components, verifying:

$$w\mathbf{V}\xi^{a,\sigma} = \rho\xi^{a,\sigma} \tag{24}$$

This is equivalent to searching the eigenvalues and eigenvectors of $\mathbf{V}$, so that:

$$\mathbf{V}u = \lambda u \tag{25}$$

and $\rho = w\lambda$. This problem is typically solved by finding the Singular Value Decomposition (SVD) of $\mathbf{B}$, and taking the normalized right singular vectors as functional principal components.

Short term forecasting using Principal Components regression

Having obtained the functional principal components, a least squares regression of the cross sections of b_is is carried out. The b_is are expressed as a linear combination of the previously constructed functional principal components, plus an error term:

$$\forall t \in \{t_1, \ldots, t_N\}\,,\; b_x^{a,\sigma}(t) = \beta_{t,0} + \sum_{k=1}^{K} \beta_{t,k}\xi_k^{a,\sigma}(x) + \epsilon_t(x) \tag{26}$$

K is the number of functional principal components. These functional principal components are not highly correlated by construction, so that we can use univariate time series forecasts for each of the $K+1$ time series, and h-step ahead forecasts of the b_is as:

$$\hat{b}_x^{a,\sigma}(t+h) = \hat{\beta}_{t+h|t,0} + \sum_{k=1}^{K} \hat{\beta}_{t+h|t,k} \xi_k^{a,\sigma}(x) \tag{27}$$

Once the forecasts $\hat{b}_x^{a,\sigma}(t+h)$ are obtained, they can be plugged into formulae 9 and 11 to deduce h-step ahead forecasts for the discount factors and discount rates.

For choosing *good* values for a, σ and K, we typically used a cross-validation on grids of values for these 3 parameters, and rolling origin estimation/forecasting, as described in section 4.

4 Numerical examples

In order to illustrate how the methods described in the previous sections work, we use IRS and OIS data from [2], [3], [1], an example of bonds data from [12]; *a curve where all cubic splines produce negative forward rates*. For forecasting the curves, we use market EUR 6M IRS data, (from which we give detailed summaries) with a CRA adjustment equal to 10bps.

For the data from [3], we assume that the swaps cash-flows paiments occur on an annual basis as for OIS. From [1], we consider mid quotes from Eonia OIS and 6-month Euribor IRS as of December 11, 2012. These data sets are all reproduced in the appendices.

In section 4.1.1, four calibration methods are tested to illustrate section 2.1. The method proposed in this paper [2] is denoted by CMN. It is compared to two iterative curve calibration methods, with linear (LIN) and natural cubic splines (SPL) interpolation on missing dates, and the [24] method (SW). Section 4.1.2 also illustrates 2.1. We use a dataset from [2]; a direct *bootstrapping* without regularization produces wiggly spot and forward rates. The effects of the regularization of approximate second derivative for forward rates and calibrated b_i's is illustrated. Such a regularization could also be applied to noisy bonds data.

In section 4.1.3, the interpolation method is tested on a *curve where all cubic methods produce negative forward rates*, from [12]. Section 4.2 illustrates the possible extrapolation methods described in section 2.2. And 4.3 illustrates the curves' forecasting methode introduced in section 3.

[2] actually applied to Hull & White model, but which can be applied to other short rates models

The discount factors usually display no particular subtleties, so they are deliberately omitted. We present discount rates and discrete forwards instead, and the discrete forwards are taken to be 3-month forward rates.

4.1 Curve calibration

4.1.1 On swaps data from [3]

Below are the discount rates and discrete forwards obtained for the 4 methods described in the previous section; two *bootstrapping* methods, with linear (LIN) and natural cubic splines (SPL) interpolation on missing dates, the [24] method (SW), and the method described in 2.1, denoted as CMN.

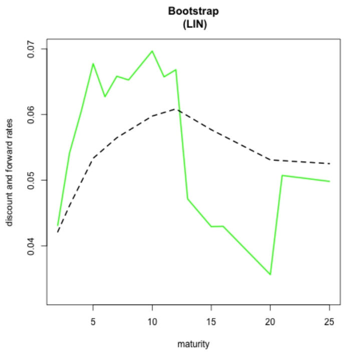

Figure 1 *Bootstrapping* with linear interpolation

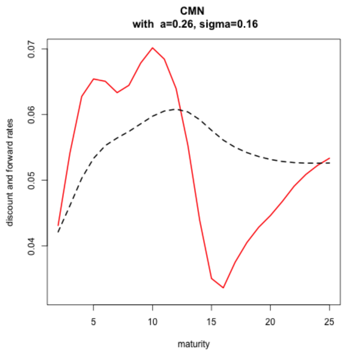

Figure 2 CMN applied to Hull and White zero coupons

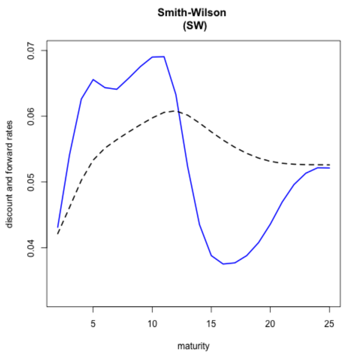

Figure 3 Smith-Wilson method

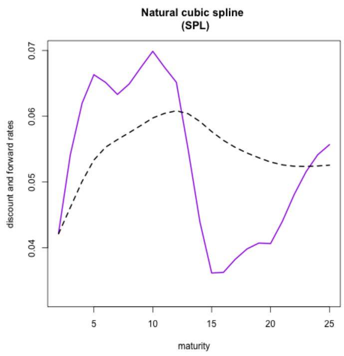

Figure 4 Natural cubic spline

Table 1 Parameters obtained for CMN and Smith-Wilson

Maturity	b_i	ξ_i
1	0.0661	-16.680
2	0.1894	23.556
3	0.2523	-0.8413
5	0.2523	-8.9116
7	0.2523	3.3552
10	0.2806	7.9600
12	0.2523	-14.098
15	0.2089	3.9119
20	0.2553	3.4828
25	0.2616	-1.9497

The discount rates are presented as a dashed line, and the forward rates as a plain colored line.

As demonstrated on figures 1, 2, 3 and 4, the discount rates produced by the 4 methods are quite similar. The discrete forward rates better exhibit the differences between them. Curve construction with linear interpolation between quoted swaps maturities (on figure 1), produces a saw-tooth like forward curve, which might not be desirable, and natural cubic spline produces the most regular discrete forwards.

For the method described in this paper (denoted as CMN on figure 2), the discrete forwards reflect the fact that the discount factors' construction relies on a piece-wise constant function, with slight changes in first derivatives at quoted swap maturities. This effect remains very reasonable however, as the discrete forward curve is highly similar to those produced by the other models, and doesn't exhibit large changes at quoted swap maturities.

For a=`0.2557`, σ=`0.1636`, the parameters b_is from table 1 are obtained. They are presented along with the parameters ξ_is obtained by the [24] method, with $a = 0.1$ (actually given as default parameter by Solvency II's technical specifications, and using the notations from QIS5 technical specifications).

4.1.2 On noisy swaps data from [2]

This section illustrates what may happen if the method from section 2.1 is applied directly to noisy data, without regularization of the parameters. We use data from [2].

Figure 5 on the left describes the discount and forward rates obtained without regularization, with randomly picked $a = 0.3655$ and $\sigma = 0.0037$. On the right, figure 6 describes the discount and forward rates obtained by minimizing the objective function in equation (15), and using the parameters $\lambda_1 = 1e - 08$ and $\lambda_2 = 1e - 05$, $a = 9.8891$ and $\sigma = 0.3957$.

In order to pick λ_1 and λ_2, we make a grid search on couples (λ_1, λ_2). For each (λ_1, λ_2), a minimization based on derivatives is applied, with multiple

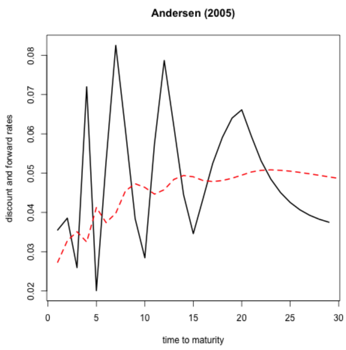

Figure 5 Curve calibration without regularization

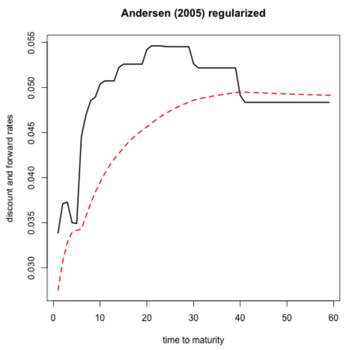

Figure 6 Curve calibration with regularization

Table 2 Parameters obtained for unregularized and regularized CMN

Maturity	unregularized b_i	regularized b_i
0.5	0.0253	0.0281
1	0.1100	0.0363
1.5	-0.0078	0.0383
2	0.0929	0.0383
2.5	-0.0005	0.0380
3	-0.1360	0.0352
4	0.2901	0.0358
5	0.1975	0.0478
7	0.1654	0.0497
10	-0.0056	0.0515
12	0.1315	0.0533
15	0.1392	0.0554
20	0.0688	0.0553
30	0.039	0.0491

restarts of the minimization algorithm. Multiple restarts avoid getting trapped into local minima.

Table 2 contains both the unregularized and regularized b_is. The unregularized ones naturally exhibit a higher variance, because an exact fit to each swap rate in the noisy dataset is required. The regularized b_is exhibit a lower variance, at the expense of a higher bias in the fitting of [2] data.

4.1.3 On a curve where all cubic methods produce negative forward rates, with data from [12]

The dataset from this section is used in [12], and is described as *a curve where all cubic methods produce negative forward rates*. It is reproduced in the appendices.

Figure 7 illustrates the discount rates (dashed line), and discrete forward rates (plain coloured line) obtained with a linear interpolation of the bond yields. The discrete forward remain positive on all maturities, but again exhibit a sawtooth profile. As expected, the natural cubic spline on figure 8 produces negative discrete forward rates on this dataset.

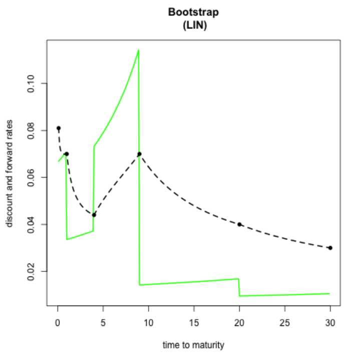

Figure 7 Linear interpolation on *a curve where all cubic methods produce negative forward rates*

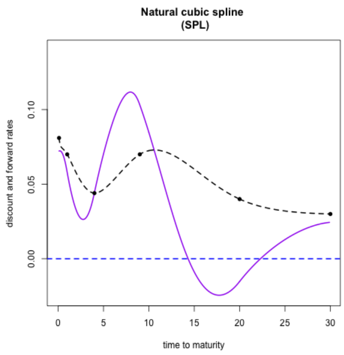

Figure 8 Natural cubic spline interpolation on *a curve where all cubic methods produce negative forward rates*

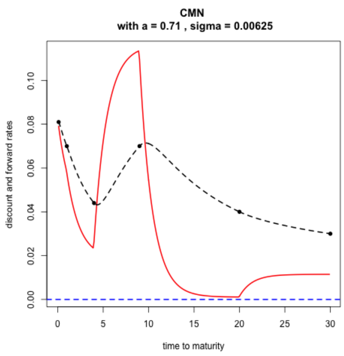

Figure 9 CMN interpolation on *a curve where all cubic methods produce negative forward rates*

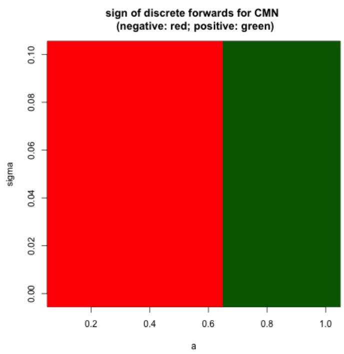

Figure 10 Sign of discrete forwards for CMN as function of a and σ, on *a curve where all cubic methods produce negative forward rates*

Table 3 Parameters obtained CMN with $a = 0.71$ and $\sigma = 0.0062$ on [12] data

Maturity	b_i
0.1	0.0718
1	0.0351
4	0.0018
9	0.1162
20	0.0011
30	0.0114

Figures 9 and 10 present the results obtained with an interpolation based on formulas (9) and (11). As seen on figure 10, a low value of a might produce negative forward rates on maturities comprised between 15 and 20. But a high value always produces positive forward rates.

This is explained by what we saw in section 2.2: in the Hull and White extend Vasicek case, a controls the speed of convergence of forward rates to the UFR. The higher the a, the faster the convergence of forward rates to the UFR on long-term maturities.

The parameters obtained by CMN interpolation (for producing figure 9), with $a = 0.71$ and $\sigma = 0.0062$ are presented in table 3.

4.2 Curve extrapolation on data from [1]

In this section, we use the extrapolation methods described in 2.2, on OIS and IRS (with CRA adjustment equal to 10bps) data from [1].

4.2.1 With Solvency II technical specifications, on IRS + CRA

Extrapolation to a fixed UFR equal to 4.2% is tested, using CMN and the Smith-Wilson method. For both methods, the Last Liquid Point (LLP) is equal to 20 years, and convergence to the UFR is forced to 40 years after the LLP.

For the CMN method, the parameters are $a = 0.174$ and $\sigma = 0.0026$, and for the Smith-Wilson method, $a = 0.125$. The resulting discount and forward curves are presented in figures 11 and 12, and the parameters b_is and ξ_is in table 4.

The discount and forward curves produced by both methods are similar, as seen on figures 11 and 12. The convergence of the Smith-Wilson method to the UFR seems to be slighty faster. This is caused by the fact that for CMN, we use instantaneous forward rates to assess the convergence to the UFR, whereas for the Smith-Wilson method, we use discrete forwards.

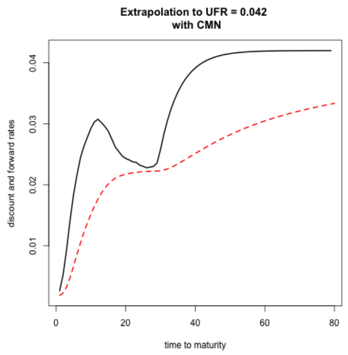

Figure 11 Extrapolation to $UFR = 4.2\%$ with CMN

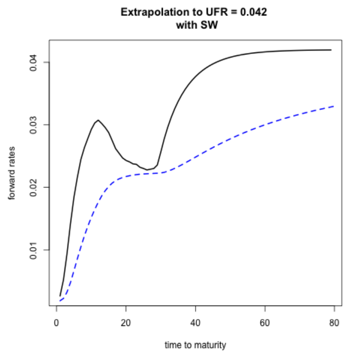

Figure 12 Extrapolation to $UFR = 4.2\%$ with Smith-Wilson

4.2.2 With OIS data, and a data driven UFR

For this example, we use OIS data from [1] presented in the appendices.

A training set containing 14 swap rates (90% of the dataset) with increasing maturities starting at 1 and ending at 20 is made up. This training set is used to construct the discount curve, which is then extrapolated to 30-year maturity and beyond, using different values for the UFR.

The 2 remaining swaps, with maturities equal to 25 and 30, are placed into the test set.

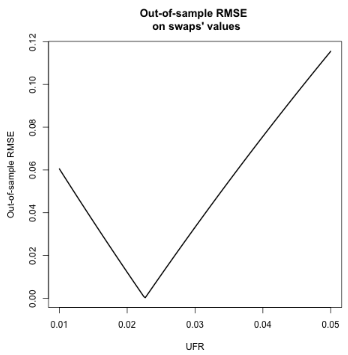

Figure 13 Out-of-sample RMSE on swap values, as a function of UFR

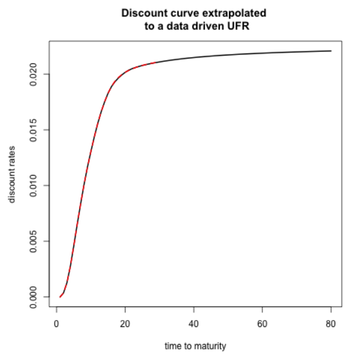

Figure 14 Extrapolation of OIS curve to a data driven $UFR = 0.0226$

Figure 13 presents the out-of-sample RMSE obtained on swaps values from the test set, as a function of UFR. This error decreases until $UFR = 0.0226$ (notice that this value would depend on the step chosen on the grid of UFRs), and then, starts to increase again. Figure 14 displays the discount curve con-

Table 4 Parameters for CMN (b_i) and Smith-Wilson (ξ_i) extrapolation

Maturity	b_i	ξ_i
1	0.0019	-2.5888
2	0.0112	0.7585
3	0.0266	0.1415
4	0.0352	1.3153
5	0.0438	0.4726
6	0.0378	-0.8809
7	0.0399	1.2010
8	0.0387	-0.8965
9	0.0338	-0.3536
10	0.0376	0.7268
11	0.0363	-0.1582
12	0.0353	1.2852
13	0.0312	-1.9866
14	0.0239	0.4161
15	0.0285	0.7056
16	0.0211	-0.7112
17	0.0208	-1.7105
18	0.0182	1.9922
19	0.0248	-1.5542
20	0.0172	0.5125
21	0.0272	1.0148
22	0.0189	-2.1158
23	0.0025	3.4051
24	0.0021	-3.7822
25	0.0020	2.7013
26	0.0239	-2.8668
27	0.0195	2.2513
28	0.0274	-0.8877
29	0.0202	-7.1463
30	0.0326	8.5322

structed on the training set, extrapolated to a 80-year maturity with an UFR equal to 0.0226 (the one minimizing the out-of-sample RMSE on the chosen grid of UFRs) is presented.

4.3 12-months ahead forecast on historical IRS + CRA

In this section, we apply ideas from section 3 to real world IRS data observed monthly from december 2013 to april 2016, adjusted from a CRA equal to 10bps.

Figure 15 and table 5 are to be read together. They contain the informations on the spot rates derived from the IRS data adjusted from a CRA, using CMN with guessed values of $a = 0.3655$ and $\sigma = 0.0037$ (other values than $a = 0.3655$ and $\sigma = 0.0037$ would produce the same results as the fitting is exact for many different values of these parameters).

Table 5 Descriptive statistics for the spot rates observed from december 2013 to april 2016

Maturity	Min.	1st Qrt	Median	Mean	3rd Qrt	Max.
1	-0.0026	-0.0008	0.0000	0.0003	0.0019	0.0031
3	-0.0023	0.0002	0.0009	0.0013	0.0028	0.0065
5	-0.0008	0.0017	0.0030	0.0037	0.0056	0.0117
10	0.0046	0.0059	0.0090	0.0101	0.0132	0.0211
15	0.0063	0.0097	0.0127	0.0141	0.0179	0.0258
20	0.0069	0.0113	0.0144	0.0157	0.0199	0.0272
30	0.0071	0.0118	0.0155	0.0164	0.0208	0.0270

The static curves are generally upward sloping, and as time passes, lower and lower spot rates are encountered. In addition, negative rates are observed in table 5; which is coherent with the current context.

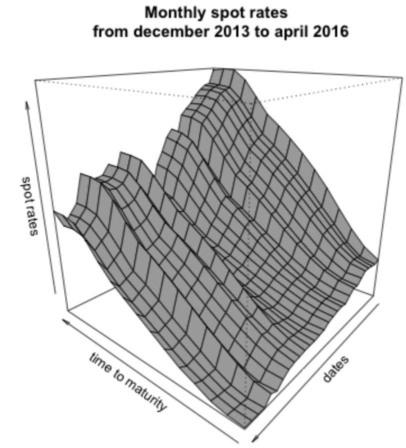

Figure 15 Spot rates observed from december 2013 to april 2016

4.3.1 Benchmarking the model

Benchmarks are subjective. The one presented in this section does not aim at showing that one method is always superior to the other. It aims at showing that the method presented in this paper produces forecasts which are (more than) reasonable, and actually close to other well-known methods forecasts (on this given dataset).

Forecasts from the model presented in section 3 are hence compared to those of two other models constructed in the spirit of by the [10]. The cross

Table 6 Average out-of-sample error on real world IRS data + CRA

Method	Parameters	Avg. OOS error
CMN - `auto.arima`	$K = 5$, $a = 1$, $\sigma = 0.1555$	0.0031
CMN - `ets`	$K = 5$, $a = 1$, $\sigma = 0.2$	0.0037
NS - `auto.arima`	$\lambda = 1.8889$	0.0031
NS - `ets`	$\lambda = 1.8889$	0.0035
NSS - `auto.arima`	$\lambda_1 = 21$, $\lambda_2 = 21$	0.0027
NSS - `ets`	$\lambda_1 = 7$, $\lambda_2 = 3$	0.0035

sections of yields described by figure 15 and table 5 are fitted by the [20] model (NS), and its extension by [25] (NSS). The formulas for the spot rates from these models are respectively:

$$R^M(t,T) = \beta_{t,1} + \beta_{t,2}\left[\frac{1-e^{-T/\lambda}}{T/\lambda}\right] + \beta_{t,3}\left[\frac{1-e^{-T/\lambda}}{T/\lambda} - e^{-T/\lambda}\right] \tag{28}$$

and

$$R^M(t,T) = \beta_{t,1} + \beta_{t,2}\left[\frac{1-e^{-T/\lambda_1}}{T/\lambda_1}\right] + \beta_{t,3}\left[\frac{1-e^{-T/\lambda_1}}{T/\lambda_1} - e^{-T/\lambda_1}\right] \tag{29}$$

$$+ \beta_{t,4}\left[\frac{1-e^{-T/\lambda_2}}{T/\lambda_2} - e^{-T/\lambda_2}\right] \tag{30}$$

Forecasts $\hat{R}^M(t+h,T)$ are obtained by fitting univariate time series to the parameters $\beta_{t,i}, i = 1,\ldots,4$ with automatic ARIMA (`auto.arima`) and exponential smoothing (`ets`) models from [17]. This automatic selection is done only for the sake of the benchmarking exercise, and in order to conduct the experience in fairly similar conditions for all the methods. **In practice, a visual inspection and an actual study of the univariate time series would of course be required**.

For all the methods the 6 methods, CMN, NS, NSS with `auto.arima` and `ets`, we obtain 12-months ahead forecasts, from rolling estimation windows of a fixed 6 months length, starting in december 2013. That is, the models are trained on 6 months data, and predictions are made on 12 months data; successively. The average out-of-sample RMSE are then calculated for each method, on the whole surface of observed and forecasted yields.

The best parameters for CMN are obtained by cross-validation, with $K \in \{2,3,4,5,6\}$, 5 values of a comprised between 0.9 and 1, and 10 values of σ comprised between 0 and 0.2. For NS and NSS, λ_1 and λ_2 are chosen by cross-validation, using the rolling estimation/forecasting we have just described.

Table 7 Importance of Principal components

Indicator	PC1	PC2	PC3
Standard deviation	0.1286	0.2461	0.2246
Proportion of variance (in %)	99.2415	0.5489	0.1315
Cumulative Proportion (in %)	99.2415	99.7904	99.9220

4.3.2 Bootstrap simulation of 12-months ahead spot rates

In this section, we use the last 12 months of the dataset to construct the functional principal components. Using 12 months as the length of the fixed window for estimation, we get an average out-of-sample RMSE of 0.0026 (on a smaller number of testing samples than the 6 months estimation window, of course).

An AR(1) is fitted to the observed univariate time series $(\beta_{t,i})_t$, $i = 0, \ldots, K$, with $a = 1$, $\sigma = 0.0089$, and $K = 3$ chosen by cross-validation. The 3 functional principal components are presented on figure 16, and some of their characteristics are summarized in table 7. We notice that the first functional principal component explains already 99.2415% of the changes in b_is, and the first 3 functional principal components selected by cross-validation explain 99.9220%.

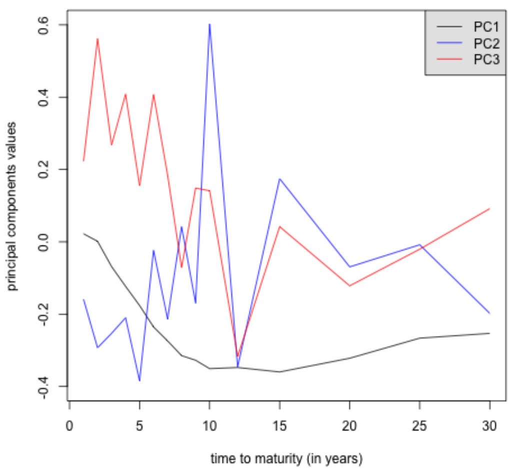

Figure 16 Principal components of the b_is from april 2015 to april 2016

Figure 17 presents the autocorrelation functions of the residuals of AR(1) fitted to $(\beta_{t,i})$, $i = 0, \ldots, 3$ from april 2015 to april 2016. The residuals from

Table 8 Descriptive statistics for fitted parameters b_is from april 2015 to april 2016

Maturity	Min.	1st Qrt	Median	Mean	3rd Qrt	Max.
1	-0.0026	-0.0021	-0.0010	-0.0013	-0.0004	-0.0003
3	0.0000	0.0026	0.0040	0.0035	0.0048	0.0058
5	0.0025	0.0076	0.0030	0.0092	0.0108	0.0143
10	0.0115	0.0174	0.0090	0.0188	0.0208	0.0230
15	0.0122	0.0168	0.0127	0.0192	0.0220	0.0228
20	0.0117	0.0148	0.0144	0.0171	0.0195	0.0211
30	0.0080	0.0116	0.0155	0.0134	0.0150	0.0178

AR(1) fitted to $(\beta_{t,i})_t$, $i = 1, \dots, 3$ could be considered as stationary, but those from the AR(1) fitted to $(\beta_{t,0})_t$ seems to be closer to an AR(4).

We denote these residuals by $(\epsilon_{t,i})_t$, $i = 0, \dots, 3$. In order to obtain simulations for the $(\beta_{t,i})_t$, $i = 0, \dots, 3$, it is possible to use a gaussian hypothesis on the residuals.

We choose to create 1000 bootstrap resamples with replacement of the $(\epsilon_{t,i})_t$, $i = 0, \dots, 3$ [3], denoted as $(\epsilon^*_{t,i})_t$, $i = 0, \dots, 3$, and create new pseudo values for $(\beta_{t,i})$, $i = 0, \dots, 3$:

$$\beta^*_{t,i} = \beta_{t,i} + \epsilon^*_{t,i},\ i = 0, \dots, 3$$

Having done this, AR(1) forecasts $\beta^*_{t+h|t,i}$ can be obtained, in order to construct:

$$\hat{b}_x^{a,\sigma,*}(t+h) = \hat{\beta}^*_{t+h|t,0} + \sum_{k=1}^{K} \hat{\beta}^*_{t+h|t,k} \xi_k^{a,\sigma}(x) \tag{31}$$

The $\hat{b}_x^{a,\sigma,*}(t+h)$ can then be plugged into formulae 9 and 11 to deduce simulations of h-step ahead forecasts for the discount factors and discount rates.

The 1000 simulations of 12-months ahead discount rates are presented in figures 18 and 19.

[3] even if for $\epsilon_{t,0}$, considering figure 17, this makes a strong stationarity assumption on the residuals

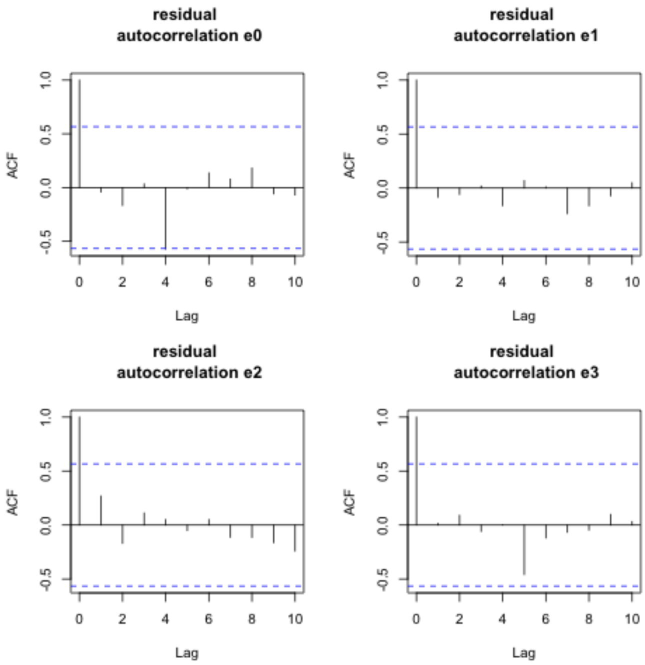

Figure 17 Autocorrelation functions for the residuals of univariate time series(AR(1)) on β_0, β_1, β_2, β_3

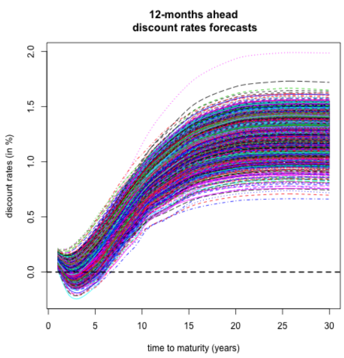

Figure 18 Curves simulated with principal components from april 2015 to april 2016, and bootstrap ressampling of the residuals

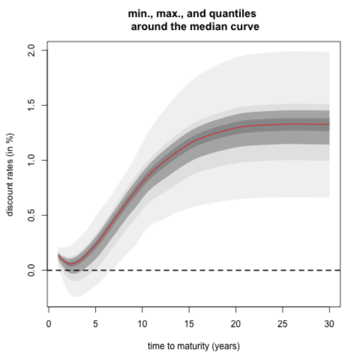

Figure 19 Min., Max., and quartiles around the median curve for the simulations

5 Conclusion

In this paper, we introduced a method for swap discount curve construction and extrapolation. This method relies on the closed form formulas for discount factors available in exogenous short rate models. We presented different ways to calibrate and extrapolate the model on different data sets from the existing literature. Moreover, we showed that the model's parameters contain a certain predictive power, enabling to obtain reasonable swap curves' forecasts, with predictive distribution.

6 Appendix

6.1 Data from [3]

Maturity	Swap Par Rate
1	4.20%
2	4.30%
3	4.70%
5	5.40%
7	5.70%
10	6.00%
12	6.10%
15	5.90%
20	5.60%
25	5.55%

6.2 Data from [2]

Maturity	Swap Par Rate
0.5	2.75%
1	3.10%
1.5	3.30%
2	3.43%
2.5	3.53%
3	3.30%
4	3.78%
5	3.95%
7	4.25%
10	4.50%
12	4.65%
15	4.78%
20	4.88%
30	4.85%

6.3 Data from [12]

Maturity	Continuous yield
0.1	8.10%
1	7.00%
4	4.40%
9	7.00%
20	4.00%
30	3.00%

6.4 Data from [1]

Maturity	EUR6M IRS	Eonia OIS
1	0.286%	0.000%
2	0.324%	0.036%
3	0.424%	0.127%
4	0.576%	0.274%
5	0.762%	0.456%
6	0.954%	0.647%
7	1.135%	0.827%
8	1.303%	0.996%
9	1.452%	1.147%
10	1.584%	1.280%
11	1.703%	1.404%
12	1.809%	1.516%
13	1.901%	-
14	1.976%	-
15	2.037%	1.764%
16	2.086%	-
17	2.123%	-
18	2.150%	-
19	2.171%	-
20	2.187%	1.939%
21	2.200%	-
22	2.211%	-
23	2.220%	-
24	2.228%	-
25	2.234%	2.003%
26	2.239%	-
27	2.243%	-
28	2.247%	-
29	2.251%	-
30	2.256%	2.038%
35	2.295%	-
40	2.348%	-
50	2.421%	-
60	2.463%	-

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