



On Multivariate Extensions of Value-at-Risk and Conditional-Tail-Expectation

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Multivariate risk problems require multivariate measures

- Many problems in finance and insurance can only be considered in a multivariate setting
- Construction of risk measures that account both for marginal effects and dependence between risks
- Multivariate risk measures involved in several applications
 - 1) Capital allocation problem
 - 2) Measures of systemic risk
 - 3) Measures for risks with heterogeneous characteristics

Measures for risk with heterogeneous characteristics

- Risks that cannot be aggregated together

- Two kinds of literature:

- 1) Extension of existing axioms to a multivariate setting

Jouini, Meddeb, Touzi (2004), Burgert and Rüschendorf (2006), Rüschendorf (2006), Cascos and Molchanov (2007), Hamel and Heyde (2010), Ekeland, Galichon, Henry (2012)

- 2) Quantile-based risk measures

Massé and Theodorescu (1994), Koltchinskii (1997), Embrechts and Puccetti (2006), Nappo and Spizzichino (2009), Prékopa (2010), Lee and Prékopa (2012)

Construction of Multivariate Risk Measures

$$\rho : \mathbf{X} := (X_1, \dots, X_d) \mapsto \begin{pmatrix} \rho^1[\mathbf{X}] \\ \vdots \\ \rho^d[\mathbf{X}] \end{pmatrix} \in \mathbb{R}_+^d,$$

Some desirable properties:

- Combine in a concise way information on both marginals and risks dependencies
- Compatible with univariate version when $d = 1$
- Easily computable for large class of multivariate distribution functions
- Consistent with usual invariance properties (Artzner et al.'s axioms)
- Consistent behavior with respect to risk perturbations

Multivariate *Value-at-Risk* based on quantile curves (Embrechts & Puccetti, 2006; Nappo & Spizzichino, 2009):

$$\partial \underline{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d : F(\mathbf{x}) = \alpha\}$$

$$\partial \bar{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d : \bar{F}(\mathbf{x}) = 1 - \alpha\}$$

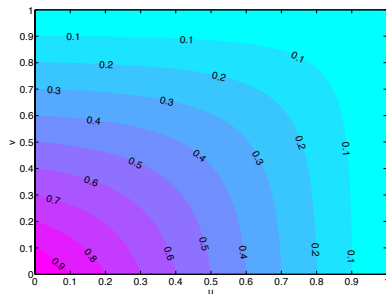
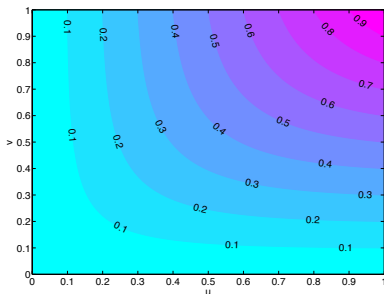


Figure: **left:** quantile curves of Frank copula with parameter 4; **right:** quantile curves of the associated survival distribution function

Lower-Orthant and Upper-Orthant **Value-at-Risk**

Definition

Consider a random vector \mathbf{X} with absolutely continuous cdf F and survival function \bar{F} . For $\alpha \in (0, 1)$, we define:

$$\underline{\text{VaR}}_{\alpha}(\mathbf{X}) := \mathbb{E}[\mathbf{X} | F(\mathbf{X}) = \alpha] = \begin{pmatrix} \mathbb{E}[X_1 | F(\mathbf{X}) = \alpha] \\ \vdots \\ \mathbb{E}[X_d | F(\mathbf{X}) = \alpha] \end{pmatrix}$$

$$\overline{\text{VaR}}_{\alpha}(\mathbf{X}) := \mathbb{E}[\mathbf{X} | \bar{F}(\mathbf{X}) = 1 - \alpha] = \begin{pmatrix} \mathbb{E}[X_1 | \bar{F}(\mathbf{X}) = 1 - \alpha] \\ \vdots \\ \mathbb{E}[X_d | \bar{F}(\mathbf{X}) = 1 - \alpha] \end{pmatrix}$$

When $d = 1$: $\underline{\text{VaR}}_{\alpha}(X) = \overline{\text{VaR}}_{\alpha}(X) = \text{VaR}_{\alpha}(X)$

Invariance Properties

Proposition

Consider a risk portfolio $\mathbf{X} = (X_1, \dots, X_d)$ and a constant vector $\mathbf{c} = (c_1, \dots, c_d)$ with positive components

- *Positive Homogeneity:*

$$\underline{\text{VaR}}_{\alpha}(c_1 X_1, \dots, c_d X_d) = (c_1 \underline{\text{VaR}}_{\alpha}^1(\mathbf{X}), \dots, c_d \underline{\text{VaR}}_{\alpha}^d(\mathbf{X}))'$$

$$\overline{\text{VaR}}_{\alpha}(c_1 X_1, \dots, c_d X_d) = (c_1 \overline{\text{VaR}}_{\alpha}^1(\mathbf{X}), \dots, c_d \overline{\text{VaR}}_{\alpha}^d(\mathbf{X}))'$$

- *Translation Invariance:*

$$\underline{\text{VaR}}_{\alpha}(\mathbf{c} + \mathbf{X}) = \mathbf{c} + \underline{\text{VaR}}_{\alpha}(\mathbf{X}), \quad \overline{\text{VaR}}_{\alpha}(\mathbf{c} + \mathbf{X}) = \mathbf{c} + \overline{\text{VaR}}_{\alpha}(\mathbf{X})$$

Comonotonic additivity

Definition (π -comonotonicity: Puccetti and Scarsini (2010))

A couple (\mathbf{X}, \mathbf{Y}) of d -dimensional random vectors is said to be π -comonotonic if there exists a d -dimensional random vector $\mathbf{Z} = (Z_1, \dots, Z_d)$ and non-decreasing functions $f_1, \dots, f_d, g_1, \dots, g_d$ such that

$$(\mathbf{X}, \mathbf{Y}) \stackrel{d}{=} (f_1(Z_1), \dots, f_d(Z_d), g_1(Z_1), \dots, g_d(Z_d)))$$

Proposition

Let (\mathbf{X}, \mathbf{Y}) be a π -comonotonic couple of random vectors, then

$$\underline{\text{VaR}}_\alpha(\mathbf{X} + \mathbf{Y}) = \underline{\text{VaR}}_\alpha(\mathbf{X}) + \underline{\text{VaR}}_\alpha(\mathbf{Y}),$$

$$\overline{\text{VaR}}_\alpha(\mathbf{X} + \mathbf{Y}) = \overline{\text{VaR}}_\alpha(\mathbf{X}) + \overline{\text{VaR}}_\alpha(\mathbf{Y})$$

Archimedean copula dependence structure

Proposition

Let \mathbf{X} be a d -dimensional portfolio of risks with marginal distributions F_1, \dots, F_d .

- If \mathbf{X} admits an **Archimedean copula** with generator ϕ , then

$$\underline{\text{VaR}}_\alpha^i(\mathbf{X}) = \mathbb{E} \left[F_i^{-1} \left(\phi^{-1}(S\phi(\alpha)) \right) \right], \quad i = 1, \dots, d$$

- If $\tilde{\mathbf{X}}$ admits an **Archimedean survival copula** with generator ϕ , then

$$\overline{\text{VaR}}_\alpha^i(\tilde{\mathbf{X}}) = \mathbb{E} \left[F_i^{-1} \left(1 - \phi^{-1}(S\phi(1 - \alpha)) \right) \right], \quad i = 1, \dots, d$$

where S is a random variable with $\text{Beta}(1, d - 1)$ distribution.

Proof : follows from [McNeil and Nešlehová \(2009\)](#) representation of Archimedean copulas

Archimedean copula dependence structure

Proposition (McNeil and Nešlehová (2009))

Let $\mathbf{U} = (U_1, \dots, U_d)$ be distributed according to a d -dimensional Archimedean copula C with generator ϕ , then

$$(\phi(U_1), \dots, \phi(U_d)) \stackrel{d}{=} R\mathbf{S},$$

where

- $\mathbf{S} = (S_1, \dots, S_d)$ is uniformly distributed on the unit simplex $\{\mathbf{x} \geq 0 \mid \sum_{k=1}^d x_k = 1\}$
- R is an independent non-negative random variable (radial part of $(\phi(U_1), \dots, \phi(U_d))$)

One can deduce that

$$1) R \stackrel{d}{=} \phi(C(\mathbf{U}))$$

$$2) [\mathbf{U} \mid C(\mathbf{U}) = \alpha] \stackrel{d}{=} (\phi^{-1}(S_1\phi(\alpha)), \dots, \phi^{-1}(S_d\phi(\alpha)))$$

Explicit expressions for Archimedean copulas

Proposition

- If \mathbf{X} is distributed as an *Archimedean copula* with generator ϕ , then

$$\underline{\text{VaR}}_{\alpha}^i(\mathbf{X}) = 1 - \int_{\alpha}^1 \left(1 - \frac{\phi(u)}{\phi(\alpha)}\right)^{d-1} du$$

- If $\tilde{\mathbf{X}}$ has uniform marginals and admits an *Archimedean survival copula* with generator ϕ , i.e. $\tilde{\mathbf{X}} \stackrel{d}{=} 1 - \mathbf{X}$, then

$$\overline{\text{VaR}}_{\alpha}^i(\tilde{\mathbf{X}}) = \int_{1-\alpha}^1 \left(1 - \frac{\phi(u)}{\phi(1-\alpha)}\right)^{d-1} du$$

Explicit expressions for bivariate Clayton copulas

Generator ϕ of a Clayton copula with dependence parameter θ :

$$\phi(u) = \frac{1}{\theta} (u^{-\theta} - 1), \quad u \in (0, 1)$$

Copula	θ	$\underline{\text{VaR}}_{\alpha, \theta}^i(X, Y)$	$\overline{\text{VaR}}_{\alpha, \theta}^i(\tilde{X}, \tilde{Y})$
Clayton C_θ	$(-1, \infty)$	$\frac{\theta}{\theta-1} \frac{\alpha^\theta - \alpha}{\alpha^{\theta-1} - 1}$	$1 - \frac{\theta}{\theta-1} \frac{(1-\alpha)^\theta - (1-\alpha)}{(1-\alpha)^{\theta-1} - 1}$
Counter-monotonic	-1	$\frac{1+\alpha}{2}$	$\frac{\alpha}{2}$
Independent	0	$\frac{\alpha-1}{\ln \alpha}$	$1 + \frac{\alpha}{\ln(1-\alpha)}$
Comonotonic	∞	α	α

Table: Components $i = 1, 2$ of $\underline{\text{VaR}}^i$ and $\overline{\text{VaR}}^i$ where (X, Y) follows a Clayton copula and $(\tilde{X}, \tilde{Y}) \stackrel{d}{=} (1 - X, 1 - Y)$, i.e., (\tilde{X}, \tilde{Y}) has a survival Clayton copula with uniform margins

Behavior of VaR components: bivariate Clayton case

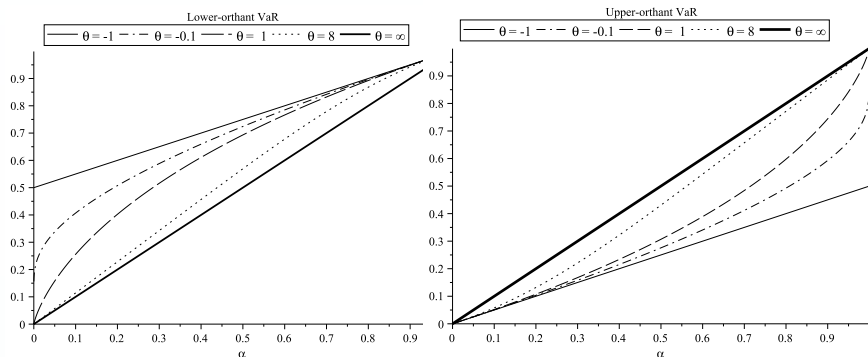


Figure: Behavior of $\underline{\text{VaR}}_{\alpha, \theta}^1(X, Y)$ (left) and $\overline{\text{VaR}}_{\alpha, \theta}^1(\tilde{X}, \tilde{Y})$ (right) with respect to risk level α and dependence parameter θ

Comparison with univariate VaR, behavior with respect to α

	$\underline{\text{VaR}}_\alpha(\mathbf{X})$	$\overline{\text{VaR}}_\alpha(\mathbf{X})$
Comparison with univariate VaR	<p>Univariate VaR is a lower bound:</p> <ul style="list-style-type: none"> • $\underline{\text{VaR}}_\alpha^i(\mathbf{X}) \geq \text{VaR}_\alpha(X_i), \forall \alpha \in (0, 1)$. <p>Comonotonic case:</p> <ul style="list-style-type: none"> • $\underline{\text{VaR}}_\alpha^i(\mathbf{X}) = \text{VaR}_\alpha(X_i), \forall \alpha \in (0, 1)$. 	<p>Univariate VaR is an upper bound:</p> <ul style="list-style-type: none"> • $\overline{\text{VaR}}_\alpha^i(\mathbf{X}) \leq \text{VaR}_\alpha(X_i), \forall \alpha \in (0, 1)$. <p>Comonotonic case:</p> <ul style="list-style-type: none"> • $\overline{\text{VaR}}_\alpha^i(\mathbf{X}) = \text{VaR}_\alpha(X_i), \forall \alpha \in (0, 1)$.
Risk level	<p>$\underline{\text{VaR}}_\alpha^i(\mathbf{X})$ is a non-decreasing function of α. (\mathbf{X} with Archimedean copulas)</p>	<p>$\overline{\text{VaR}}_\alpha^i(\tilde{\mathbf{X}})$ is a non-decreasing function of α. ($\tilde{\mathbf{X}}$ with Archimedean survival copulas)</p>

Effect of risk perturbations

	$\underline{\text{VaR}}_\alpha(\mathbf{X})$	$\overline{\text{VaR}}_\alpha(\mathbf{X})$
Change in marginals	<p>For a fixed copula C, if $X_i \stackrel{d}{=} Y_i$:</p> <ul style="list-style-type: none"> • $\underline{\text{VaR}}_\alpha^i(\mathbf{X}) = \underline{\text{VaR}}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. <p>For a fixed copula C, if $X_i \leq_{st} Y_i$:</p> <ul style="list-style-type: none"> • $\underline{\text{VaR}}_\alpha^i(\mathbf{X}) \leq \underline{\text{VaR}}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. 	<p>For a fixed copula C, if $X_i \stackrel{d}{=} Y_i$:</p> <ul style="list-style-type: none"> • $\overline{\text{VaR}}_\alpha^i(\mathbf{X}) = \overline{\text{VaR}}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. <p>For a fixed copula C, if $X_i \leq_{st} Y_i$:</p> <ul style="list-style-type: none"> • $\overline{\text{VaR}}_\alpha^i(\mathbf{X}) \leq \overline{\text{VaR}}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$.
Change in dependence structure	<p>For fixed marginals, if $\theta \leq \theta^*$:</p> <ul style="list-style-type: none"> • $\underline{\text{VaR}}_\alpha^i(\mathbf{X}) \leq \underline{\text{VaR}}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. <p>$\mathbf{X}$ with Archimedean copula</p>	<p>For fixed marginals, if $\theta \leq \theta^*$:</p> <ul style="list-style-type: none"> • $\overline{\text{VaR}}_\alpha^i(\mathbf{X}) \leq \overline{\text{VaR}}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. <p>$\mathbf{X}$ with Archimedean survival copula</p>

Multivariate *CTE*-s based on upper-level set of multivariate cdf and lower-level set of survival functions:

$$\underline{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d : F(\mathbf{x}) \geq \alpha\}$$

$$\bar{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d : \bar{F}(\mathbf{x}) \leq 1 - \alpha\}$$

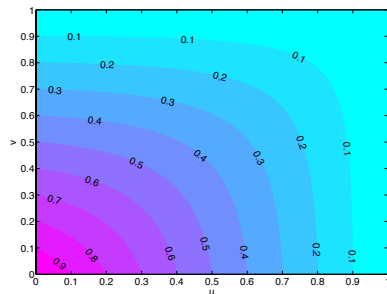
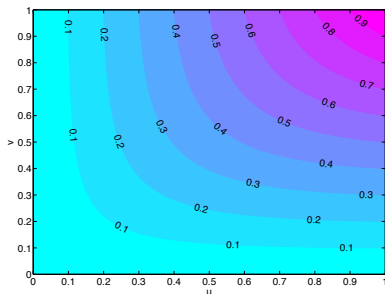


Figure: **left:** quantile curves of Frank copula with parameter 4; **right:** quantile curves of the associated survival distribution function

Lower-Orthant and Upper-Orthant **Conditional-Tail-Expectation**

Definition

Consider a random vector \mathbf{X} with absolutely continuous cdf F and survival function \bar{F} . For $\alpha \in (0, 1)$, we define:

$$\underline{\text{CTE}}_{\alpha}(\mathbf{X}) := \mathbb{E}[\mathbf{X} | F(\mathbf{X}) \geq \alpha] = \begin{pmatrix} \mathbb{E}[X_1 | F(\mathbf{X}) \geq \alpha] \\ \vdots \\ \mathbb{E}[X_d | F(\mathbf{X}) \geq \alpha] \end{pmatrix}$$

$$\overline{\text{CTE}}_{\alpha}(\mathbf{X}) := \mathbb{E}[\mathbf{X} | \bar{F}(\mathbf{X}) \leq 1 - \alpha] = \begin{pmatrix} \mathbb{E}[X_1 | \bar{F}(\mathbf{X}) \leq 1 - \alpha] \\ \vdots \\ \mathbb{E}[X_d | \bar{F}(\mathbf{X}) \leq 1 - \alpha] \end{pmatrix}$$

When $d = 1$: $\underline{\text{CTE}}_{\alpha}(X) = \overline{\text{CTE}}_{\alpha}(X) = \text{CTE}_{\alpha}(X)$

Invariance Properties

- Positive Homogeneity: $\forall \mathbf{c} \in \mathbb{R}_+^d$,

$$\underline{\text{CTE}}_\alpha(\mathbf{c}\mathbf{X}) = \mathbf{c}\underline{\text{CTE}}_\alpha(\mathbf{X}), \quad \overline{\text{CTE}}_\alpha(\mathbf{c}\mathbf{X}) = \mathbf{c}\overline{\text{CTE}}_\alpha(\mathbf{X})$$

- Translation Invariance: $\forall \mathbf{c} \in \mathbb{R}_+^d$,

$$\underline{\text{CTE}}_\alpha(\mathbf{c} + \mathbf{X}) = \mathbf{c} + \underline{\text{CTE}}_\alpha(\mathbf{X}), \quad \overline{\text{CTE}}_\alpha(\mathbf{c} + \mathbf{X}) = \mathbf{c} + \overline{\text{CTE}}_\alpha(\mathbf{X})$$

- π -comonotonic additivity: if (\mathbf{X}, \mathbf{Y}) is π -comonotonic, then

$$\underline{\text{CTE}}_\alpha(\mathbf{X} + \mathbf{Y}) = \underline{\text{CTE}}_\alpha(\mathbf{X}) + \underline{\text{CTE}}_\alpha(\mathbf{Y}),$$

$$\overline{\text{CTE}}_\alpha(\mathbf{X} + \mathbf{Y}) = \overline{\text{CTE}}_\alpha(\mathbf{X}) + \overline{\text{CTE}}_\alpha(\mathbf{Y})$$

- For $\alpha = 0$, $\underline{\text{CTE}}_0(\mathbf{X}) = \overline{\text{CTE}}_0(\mathbf{X}) = \mathbb{E}[\mathbf{X}]$

Link with multivariate VaR-s

Proposition

Under the regularity assumption, the following integral representations hold

$$\underline{CTE}_{\alpha}^i(\mathbf{X}) = \frac{1}{1 - K(\alpha)} \int_{\alpha}^1 \underline{VaR}_{\gamma}^i(\mathbf{X}) K'(\gamma) d\gamma,$$

$$\overline{CTE}_{\alpha}^j(\mathbf{X}) = \frac{1}{\widehat{K}(1 - \alpha)} \int_{\alpha}^1 \overline{VaR}_{\gamma}^j(\mathbf{X}) \widehat{K}'(1 - \gamma) d\gamma,$$

where

- K is the Kendall distribution of \mathbf{X} : $K(x) = \mathbb{P}(F(\mathbf{X}) \leq x)$, for all x in $(0, 1)$
- \widehat{K} is the upper-orthant Kendall distribution of \mathbf{X} : $\widehat{K}(x) = \mathbb{P}(\overline{F}(\mathbf{X}) \leq x)$, for all x in $(0, 1)$.

Explicit expressions for bivariate Clayton copulas

Copula	θ	$\underline{\text{CTE}}_{\alpha, \theta}^i(X, Y)$
Clayton C_θ	$(-1, \infty)$	$\frac{1}{2} \frac{\theta}{\theta-1} \frac{\theta-1-\alpha^2(1+\theta)+2\alpha^{1+\theta}}{\theta-\alpha(1+\theta)+\alpha^{1+\theta}}$
Counter-monotonic	-1	$\frac{1}{4} \frac{1-\alpha^2+2 \ln \alpha}{1-\alpha+\ln \alpha}$
Independent	0	$\frac{1}{2} \frac{(1-\alpha)^2}{1-\alpha+\alpha \ln \alpha}$
Comonotonic	∞	$\frac{1+\alpha}{2}$

Table: Components $i = 1, 2$ of $\underline{\text{CTE}}^i$ for different copula dependence structures.

Interestingly, one can readily show that $\frac{\partial \underline{\text{CTE}}_{\alpha, \theta}^i}{\partial \alpha} \geq 0$ and $\frac{\partial \underline{\text{CTE}}_{\alpha, \theta}^i}{\partial \theta} \leq 0$, for $\theta \geq -1$ and $\alpha \in (0, 1)$.

Behavior of CTE components: Clayton copula case

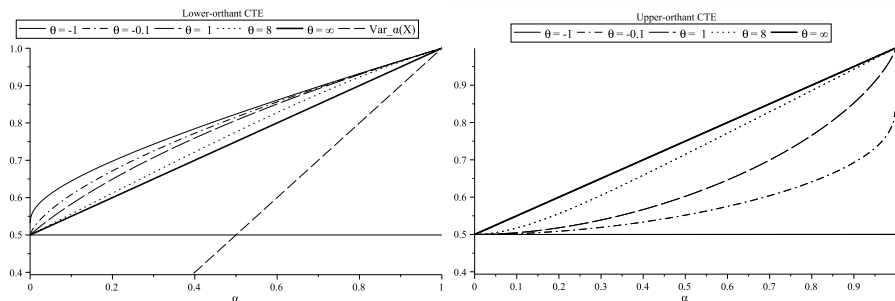


Figure: Behavior of $\underline{\text{CTE}}_{\alpha, \theta}^i(X, Y)$ (left) and $\overline{\text{CTE}}_{\alpha, \theta}^i(1 - X, 1 - Y)$ (right) with respect to risk level α for different values of dependence parameter θ .

Comonotonic case, behavior with respect to α , comparison with multivariate VaR

	$\underline{\text{CTE}}_{\alpha}(\mathbf{X})$	$\overline{\text{CTE}}_{\alpha}(\mathbf{X})$
	<p>Comonotonic case:</p> <ul style="list-style-type: none"> • $\underline{\text{CTE}}_{\alpha}^i(\mathbf{X}) = \text{CTE}_{\alpha}(X_i), \forall \alpha \in (0, 1)$. 	<p>Comonotonic case:</p> <ul style="list-style-type: none"> • $\overline{\text{CTE}}_{\alpha}^i(\mathbf{X}) = \text{CTE}_{\alpha}(X_i), \forall \alpha \in (0, 1)$.
Risk level	<ul style="list-style-type: none"> • $\underline{\text{CTE}}_{\alpha}^i(\mathbf{X})$ is a non-decreasing function of α. (\mathbf{X} with Archimedean copulas) • $\underline{\text{CTE}}_{\alpha}^i(\mathbf{X}) \geq \underline{\text{VaR}}_{\alpha}^i(\mathbf{X}), \quad \text{for all } \alpha \in (0, 1)$. 	<ul style="list-style-type: none"> • $\overline{\text{CTE}}_{\alpha}^i(\tilde{\mathbf{X}})$ is a non-decreasing function of α. ($\tilde{\mathbf{X}}$ with Archimedean survival copulas) • $\overline{\text{CTE}}_{\alpha}^i(\mathbf{X}) \geq \overline{\text{VaR}}_{\alpha}^i(\mathbf{X}), \quad \text{for all } \alpha \in (0, 1)$.

Effect of risk perturbations

	$\underline{CTE}_\alpha(\mathbf{X})$	$\overline{CTE}_\alpha(\mathbf{X})$
Change in marginals	<p>For a fixed copula C, if $X_i \stackrel{d}{=} Y_i$:</p> <ul style="list-style-type: none"> • $\underline{CTE}_\alpha^i(\mathbf{X}) = \underline{CTE}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. <p>For a fixed copula C, if $X_i \leq_{st} Y_i$:</p> <ul style="list-style-type: none"> • $\underline{CTE}_\alpha^i(\mathbf{X}) \leq \underline{CTE}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. 	<p>For a fixed copula C, if $X_i \stackrel{d}{=} Y_i$:</p> <ul style="list-style-type: none"> • $\overline{CTE}_\alpha^i(\mathbf{X}) = \overline{CTE}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. <p>For a fixed copula C, if $X_i \leq_{st} Y_i$:</p> <ul style="list-style-type: none"> • $\overline{CTE}_\alpha^i(\mathbf{X}) \leq \overline{CTE}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$.
Change in dependence structure	<p>For fixed marginals, if $\theta \leq \theta^*$:</p> <ul style="list-style-type: none"> • $\underline{CTE}_\alpha^i(\mathbf{X}) \leq \underline{CTE}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. <p>$\mathbf{X}$ with Archimedean copula</p>	<p>For fixed marginals, if $\theta \leq \theta^*$:</p> <ul style="list-style-type: none"> • $\overline{CTE}_\alpha^i(\mathbf{X}) \leq \overline{CTE}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. <p>$\tilde{\mathbf{X}}$ with Archimedean survival copula</p>

Perspectives

- Subadditivity of the proposed multivariate CTE-s ?
- Allocation problems for portfolios of portfolios
- Other perspectives:
 - Extension to discrete distribution functions
 - Comparison with other multivariate risk measures

Thank you for your attention

References

- Acharya, Pedersen, Philippon, Richardson, 2010, *Measuring systemic risk*
- Adrian and Brunnermeier, 2011, *CoVaR*
- Bargès, Cossette, Marceau, 2009, *TVaR-based capital allocation with copulas*
- Brownlees and Engle, 2012, *Volatility, correlation and tails for systemic risk measurement*
- Burgert and Rüschendorf, 2006, *Consistent risk measures for portfolio vectors*
- Cai, Einmahl, Haan, Zhou, 2013, *Estimation of the Marginal Expected Shotfall: the mean when a related variable is extreme*
- Cai and Li, 2005, *Conditional tail expectations for multivariate phase-type distributions*
- Cascos and Molchanov, 2007, *Multivariate risks and depth-trimmed regions*

References

- Ekeland, Galichon, Henry, 2012, *Comonotonic measures of multivariate risks*
- Embrechts and Puccetti, 2006, *Bounds for functions of multivariate risks*
- Hamel and Heyde, 2010, *Duality for set-valued measures of risk*
- Jouini, Meddeb, Touzi, 2004, *Vector-valued coherent risk measures*
- Koltchinskii, 1997, *M-estimation, convexity and quantiles*
- Landsman and Valdez, 2003, *Tail conditional expectations for elliptical distributions*
- Lee and Prékopa, 2012, *Properties and calculation of multivariate risk measures: MVaR and MCVaR*
- Mainik and Schaanning, 2012, *On dependence consistency of CoVaR and some other systemic risk measures*

References

- [Massé and Theodorescu](#), 1994, *Halfplane trimming for bivariate distributions*
- [McNeil and Nešlehová](#), 2009, *Multivariate Archimedean copulas, D-monotone functions and l_1 -norm symmetric distributions*
- [Nappo and Spizzichino](#), 2009, *Kendall distributions and levels sets in bivariate exchangeable survival models*
- [Prékopa](#), 2010, *Multivariate value at risk and related topics*
- [Puccetti and Scarsini](#), 2010, *Multivariate comonotonicity*
- [Rüschendorf](#), 2006, *Law invariant convex risk measures for portfolio vectors*
- [Scaillet](#), 2004, *Nonparametric estimation and sensitivity analysis of expected shortfall*
- [Tasche](#), 2008, *Capital allocation to business units and sub-portfolios: the Euler Principle*