# On the range of admissible term-structures

Areski Cousin
ISFA, Université Lyon 1

Joint work with Ibrahima Niang

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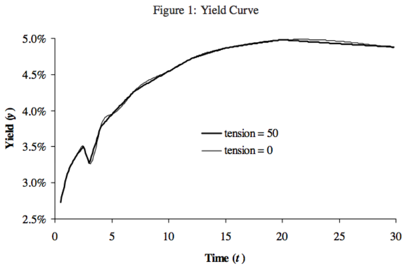
# Introduction

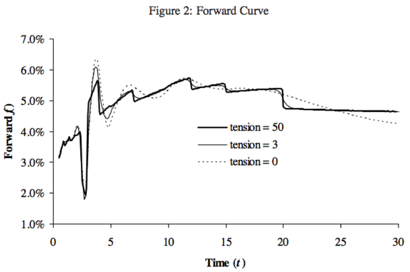
- What is understood as a term-structure in this presentation ?
- What is it used for ?

# Introduction

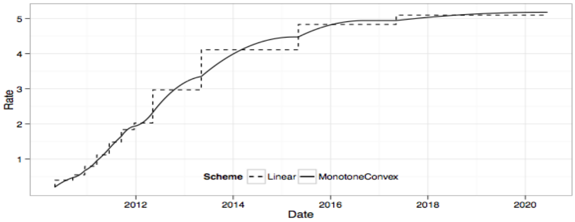
- Term-structures are constructed from market quotes of fixed income, fx or default-risky securities
- Information provided by the market is reliable only for a small set of liquid products with standard characteristics/maturities
- We have to rely on interpolation/calibration schemes to construct the curve for missing maturities
- A variety of curve construction methods exists : no consensus towards a particular best practice in all circumstances

Andersen (2007), curves based on tension splines




Le Floc'h (2012), examples of one-day forward curves



**What can be defined as a good yield curve construction method ?** (Hagan and West (2006))

- Ability to fit market quotes
- Arbitrage freeness
- Smoothness
- Locality of the interpolation method
- Stability of forward rate
- Consistency of hedging strategies : Locality of deltas ? Sum of sequential deltas close enough to the corresponding parallel delta ? (Le Floc'h (2012))

**Interestingly, there is a pretty large recent literature on the subject of yield-curve construction methods**

- Single-curve environment : Hagan and West (2006), Andersen (2007), Jerassy-Etzion (2010), Le Floc'h (2012)
- Multi-curve environment : Ametrano and Bianchetti (2009), Iwashita (2013), Kenyon and Stamm (2012), Fries (2013), Chibane et al (2009)

**And a flourishing literature on model risk**

- Branger and Schlag (2004), Cont (2006), Davis and Hobson (2004), Derman (1996), Eberlein and Jacod (1997), El Karoui et al (1998), Green and Figlewski (1999), Hénaff (2010), Morini (2010), etc...

### Arbitrage-free curve

A curve is said to be arbitrage-free if

- IR curves : the forward rates are non-negative or equivalently, the (pseudo) discount factors are nonincreasing with respect to time-to-maturities
- Credit : the curve is associated with a well-defined default distribution function

### Smoothness condition

A curve is said to be smooth if

- IR curves : the instantaneous forward rates exist for all maturities and are continuous.
- Credit : the default density function exists and is continuous.

### Admissible curve

A yield curve is said to be admissible if it satisfies the following constraints :

- The input data set is perfectly reproduced by the curve
- The curve is arbitrage-free
- The curve satisfies the smoothness condition

**We then address the following questions :**

- Is it possible to estimate the size of admissible curves ? and how ?
- How does the range/diversity of admissible curves affect the present value of products with non-standard characteristics ?

We develop a framework in which it is possible to measure the diversity of yield curves with some specific features.

# Instruments used for curve construction

**Assumption I : pseudo-linear representation of present values**

Products used in the curve construction have presents values that can be expressed as linear combination of some elementary quantities such as zero-coupon prices, discount factors, Ibor forward rates or survival probabilities.

**Example 1 : Corporate or sovereign debt yield curve**

- $S$ : market price (in percentage of nominal) at time $t_0$ of a bond with maturity $T$
- $c$ : fixed coupon rate
- $t_1 < \ldots < t_p = T$ : coupon payment dates, $\delta_k$ : year fraction corresponding to period $(t_{k-1}, t_k)$

$$c \sum_{k=1}^{p} \delta_k P^B(t_0, t_k) + P^B(t_0, T) = S$$

where $P^B(t_0, t_k)$ represents the price of a (fictitious default-free issuer-dependent) ZC bond with maturity $t_k$

**Example 2 : Discounting curve based on OIS**

- $S^{\text{OIS}}$ : par rate at time $t_0$ of an overnight indexed swap with maturity $T$
- $t_1 < \cdots < t_p = T$ : fixed-leg payment dates
- $\delta_k$ : year fraction corresponding to period $(t_{k-1}, t_k)$

$$S^{\text{OIS}} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) = 1 - P^D(t_0, T)$$

where $P^D(t_0, t_k)$ is the discount factor associated with maturity date $t_k$

**Example 3 : Forward curve based on fixed-vs-Ibor-floating IRS**

- $S^{\mathrm{IRS}}$ : par rate at time $t_0$ of an interest rate swap with maturity $T$ and tenor $j$ (typically, $j = 3\mathrm{M}$ or $j = 6\mathrm{M}$)
- $t_1 < \cdots < t_p = T$ : fixed-leg payment dates, $\delta_k$ : year fraction corresponding to period $(t_{k-1}, t_k)$
- $t = \tilde{t}_0 < \tilde{t}_1 < \cdots < \tilde{t}_q = T$ : floating-leg payment dates, $\tilde{\delta}_i$ : year fraction of $(\tilde{t}_{i-1}, \tilde{t}_i)$

$$S^{\mathrm{IRS}} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) = \sum_{i=1}^{q} P^D(t_0, \tilde{t}_i)\tilde{\delta}_i F_j(t_0, \tilde{t}_i)$$

where $F_j(t_0, \tilde{t}_i)$ is the forward Libor or Euribor rate defined as the fixed rate to be exchanged at time $\tilde{t}_i$ against the $j$-tenor Libor or Euribor rate established at time $\tilde{t}_{i-1}$ so that the swap has zero value at time $t_0$

**Example 4 : credit curve based on CDS**

- $S^{\text{CDS}}$ : fair spread at time $t_0$ of a credit default swap with maturity $T$
- $t_1 < \cdots < t_p = T$ : premium payment dates, $\delta_k$ : year fraction corresponding to period $(t_{k-1}, t_k)$
- $R$ : expected recovery rate of the reference entity

$$S^{\text{CDS}} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) Q(t_0, t_k) = -(1 - R) \int_{t_0}^{T} P^D(t_0, u) dQ(t_0, u)$$

where $u \to Q(t_0, u)$ is the $\mathcal{F}_{t_0}$-conditional (risk-neutral) survival distribution of the reference entity.

We implicitly assume here that recovery, default and interest rates are stochastically independent.

**Example 4 : credit curve based on CDS (cont)**

Using an integration by parts, the survival function $u \to Q(t_0, u)$ satisfies a linear relation :

$$S^{\text{CDS}} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) Q(t_0, t_k) + (1 - R) P^D(t_0, T) Q(t_0, T)$$
$$+ (1 - R) \int_{t_0}^{T} f^D(t_0, u) P^D(t_0, u) Q(t_0, u) du = 1 - R$$

where $f^D(t_0, u)$ is the instantaneous forward (discount) rate associated with maturity date $u$.

# Geometric nature of the problem

**Proposition (admissible curves form a convex set)**

Under Assumption I, the set of admissible yield-curve is convex.

This derives immediately from the definition of admissible curves and the linear representation of present values.

**Proposition**

Under Assumption I, the set of admissible yield-curves is characterized by the convex hull of the extreme points of its closure.

Identifying the set of admissible yield-curves amounts to identify its convex hull

# Geometric nature of the problem

The proof follows from successive applications of Ascoli-Arzelà theorem and Krein-Milman theorem.

**Ascoli-Arzelà theorem**

Let $(X, d)$ be a compact space. A subset $F$ of $\mathcal{C}(X)$ is relatively compact if and only if $F$ is equibounded and equicontinuous.

We have to prove that $F$ is equibounded and equicontinuous.

**Krein-Milman theorem**

Let $X$ be a locally convex topological vector space (assumed to be Hausdorff or separable), and let $K$ be a compact convex subset of $X$. Then $K$ is the closed convex hull of its extreme points.

# Arbitrage-free bounds for OIS discount curves

- We observe OIS par rates $S_1, \cdots, S_n$ for maturities $T_1 < \cdots < T_n$.
- Let $t = t_0 < t_1 < \cdots < t_{p_n} = T_n$ be the annual time grid up to time $T_n$.
- The set of indices $(p_i)$ is such that $t_{p_i} = T_i$ for $i = 1, ..., n$.

$$S_i \sum_{k=1}^{p_i - 1} \delta_k P^D(t_0, t_k) + (S_i \delta_{p_i} + 1) P^D(t_0, T_i) = 1, \quad i = 1, ..., n$$

- Let $i_0$ be the smallest index such that $T_{i_0} \neq t_{i_0}$ ($i_0 = 11$ in our applications)
- Define $H_i := \sum_{k=p_{i-1}+1}^{p_i - 1} \delta_k$, for $i = i_0, \ldots, n$

# Arbitrage-free bounds for OIS discount curves

**Proposition (arbitrage-free bounds for discount factors)**

$$P^D(t_0, T_1) = \frac{1}{1 + S_1\delta_1},$$

$$P^D(t_0, T_i) = \frac{1}{1 + S_i\delta_i}\left(1 - \frac{S_i}{S_{i-1}}\left(1 - P^D(t_0, T_{i-1})\right)\right), \quad i = 2, \ldots, i_0 - 1$$

For $i = i_0, \ldots, n,$

$$P^D_{\min}(t_0, T_i) \leqslant P^D(t_0, T_i) \leqslant P^D_{\max}(t_0, T_i)$$

where

$$P^D_{\min}(t_0, T_i) = \frac{1}{1 + S_i\delta_{p_i}}\left(1 - \frac{S_i}{S_{i-1}}\left(1 - (1 - S_{i-1}H_i)P^D(t_0, T_{i-1})\right)\right)$$

$$P^D_{\max}(t_0, T_i) = \frac{1}{1 + S_i(H_i + \delta_{p_i})}\left(1 - \frac{S_i}{S_{i-1}}\left(1 - P^D(t_0, T_{i-1})\right)\right)$$

**Proof :**

For any $i = i_0, \ldots, n$, the previous rectangular system of OIS present values can be simplified :

$$\frac{S_i}{S_{i-1}}\left(1 - P^D(t_0, T_{i-1})\right) + S_i \sum_{k=p_{i-1}+1}^{p_i - 1} \delta_k P^D(t_0, t_k) + (1 + S_i \delta_{p_i})\, P^D(t_0, T_i) = 1$$

The bounds derive from the following system of arbitrage-free inequalities :

$$\begin{cases} P^D(t_0, T_{i_0}) \leqslant P^D(t_0, t_k) \leqslant P^D(t_0, T_{i_0 - 1}) & \text{for } p_{i_0 - 1} + 1 \leqslant k \leqslant p_{i_0} - \\ \vdots & \\ P^D(t_0, T_i) \leqslant P^D(t_0, t_k) \leqslant P^D(t_0, T_{i-1}) & \text{for } p_{i-1} + 1 \leqslant k \leqslant p_i - 1 \end{cases}$$

These bounds cannot be computed since we do not know the discount factors $P^D(t_0, T_i)$ for $i = i_0, \ldots, n$

**Iterative computation of model-free bounds**

- **Step 1** : For $i = 1, \ldots, i_0 - 1$,

$$P^D(t_0, T_i) = \frac{1}{1 + S_i \delta_i} \left( 1 - \frac{S_i}{S_{i-1}} \left( 1 - P^D(t_0, T_{i-1}) \right) \right)$$

- **Step 2** : For $i = i_0, \ldots, n$,

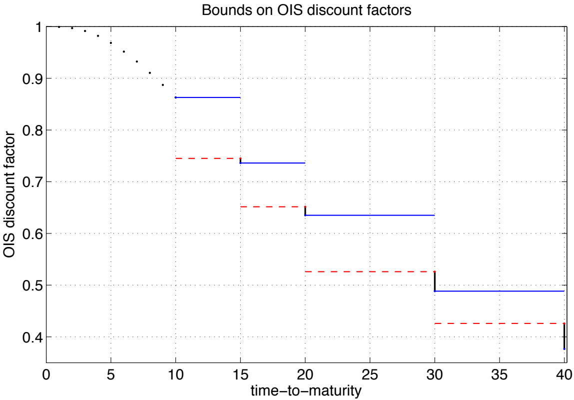
$$P_{\min}(T_i) \leqslant P^D(t_0, T_i) \leqslant P_{\max}(T_i)$$

where

$$P_{\min}(T_i) = \frac{1}{1 + S_i \delta_{p_i}} \left( 1 - \frac{S_i}{S_{i-1}} (1 - (1 - S_{i-1} H_i) P_{\min}(T_{i-1})) \right)$$

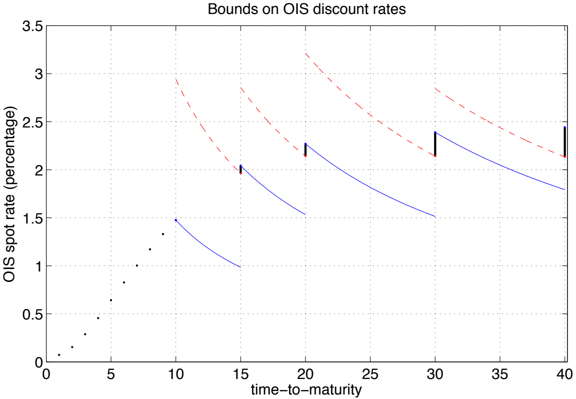
$$P_{\max}(T_i) = \frac{1}{1 + S_i (H_i + \delta_{p_i})} \left( 1 - \frac{S_i}{S_{i-1}} (1 - P_{\max}(T_{i-1})) \right)$$

**The previous model-free bounds are sharp**



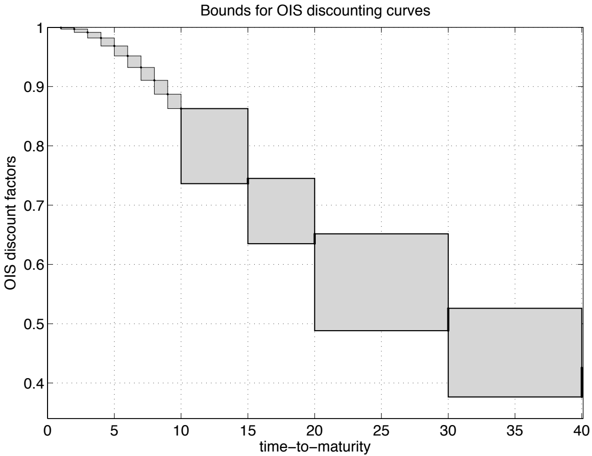

**Input data** : OIS swap rates as of May, 31st 2013

**Corresponding model-free bounds on discount rates**




**Input data** : OIS swap rates as of May, 31st 2013

**Range of arbitrage-free market-consistent OIS discount curves**




**Input data** : OIS swap rates as of May, 31st 2013

# Arbitrage-free bounds for OIS discount curves

**Proposition (detecting arbitrage opportunities)**

An arbitrage opportunity can be detected in the data set $(S_i)_{i=1,\dots,n}$ at the first index $i$ such that

$$S_i < \left( \frac{1}{S_{i-1}} + \delta_i \frac{P^D(t_0, T_{i-1})}{1 - P^D(t_0, T_{i-1})} \right)^{-1}, \; i = 2, \dots, i_0 - 1,$$

$$S_i < \left( \frac{1}{S_{i-1}} + (H_i + \delta_{p_i}) \frac{P_{\max}(T_{i-1})}{1 - P_{\max}(T_{i-1})} \right)^{-1}, \; i = i_0, \dots, n.$$

**Proof :**

For $i = 2, \dots, i_0 - 1$, the inequality on $S_i$ leads to $P^D(t_0, T_i) > P^D(t_0, T_{i-1})$

For $i = i_0, \dots, n$, the inequality on $S_i$ leads to $P^D_{\min}(t_0, T_i) > P^D_{\max}(t_0, T_i)$

**Corollary (increasing OIS par rates are arbitrage-free)**

An increasing sequence of OIS par rates $S_1 \leq \cdots \leq S_n$ is arbitrage-free : there always exits an arbitrage-free discount curve which is compatible with this sequence.

# Arbitrage-free bounds for survival curves

- We observe CDS fair spreads $S_1, \ldots, S_n$ for maturities $T_1 < \cdots < T_n$.
- Let $t = t_0 < t_1 < \cdots < t_{p_n} = T_n$ be the time grid corresponding to premium payment dates.
- The set of indices $(p_i)$ is such that $p_0 = 1$ and $t_{p_i} = T_i$ for $i = 1, ..., n$.
- For $i = 1, \ldots, n$,

$$S_i \sum_{k=1}^{p_i} \delta_k P^D(t_0, t_k) Q(t_0, t_k) + (1-R) P^D(t_0, T) Q(t_0, T)$$
$$+ (1-R) \int_{t_0}^{T_i} f^D(t_0, t) P^D(t_0, t) Q(t_0, t) dt = 1 - R$$

# Arbitrage-free bounds for survival curves

**Proposition (arbitrage-free bounds for survival probabilities)**

For $i = 1, \ldots, n$,

$$Q_{\min}(t_0, T_i) \leqslant Q(t_0, T_i) \leqslant Q_{\max}(t_0, T_i)$$

where

$$Q_{\min}(t_0, T_i) = \frac{1 - R - \sum_{k=1}^{i} \left((1-R)M_k + S_i N_k\right) Q(t_0, T_{k-1})}{P^D(t_0, T_i)(1 - R + S_i \delta_{p_i})},$$

$$Q_{\max}(t_0, T_i) = \frac{1 - R - \sum_{k=1}^{i-1} \left((1-R)M_k + S_i N_k\right) Q(t_0, T_k)}{P^D(t_0, T_{i-1})(1 - R) + S_i \left(N_i + \delta_{p_i} P^D(t_0, T_i)\right)},$$

with $M_i := P^D(t_0, T_{i-1}) - P^D(t_0, T_i)$ and $N_i := \sum_{k=p_{i-1}}^{p_i - 1} \delta_k P^D(t_0, t_k)$.

**Proof :**

For any $i = 1, \ldots, n$, the proof is based on CDS present value representations as linear combinations of survival probabilities and application of the following system of "arbitrage-free inequalities" :

$$\begin{cases} Q(t_0, T_1) \leqslant Q(t_0, t) \leqslant 1 & \text{for } t_0 \leqslant t < T_1, \\ \vdots & \\ Q(t_0, T_i) \leqslant Q(t_0, t) \leqslant Q(t_0, T_{i-1}) & \text{for } T_{i-1} \leqslant t < T_i \end{cases}$$

These bounds cannot be computed explicitly since we do not know the survival probabilities $Q(t, T_i)$ with certainty for $i = 1, \ldots, n$

**Iterative computation of model-free bounds**

- For $i = 1, \dots, n$, compute recursively

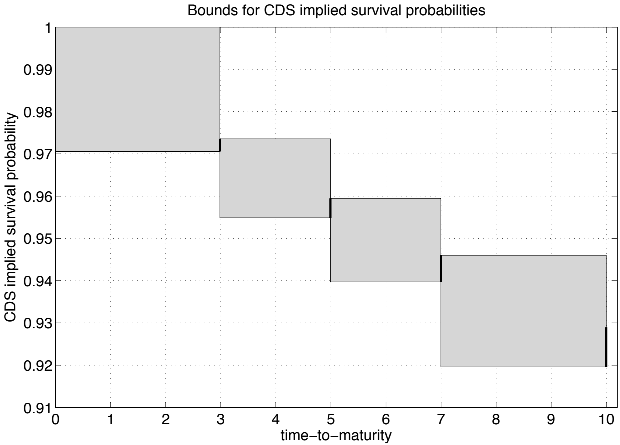
$$Q_{\min}(T_i) \leqslant Q(t_0, T_i) \leqslant Q_{\max}(T_i)$$

where

$$Q_{\min}(T_i) = \frac{1 - R - \sum_{k=1}^{i} \left((1-R)M_k + S_i N_k\right) Q_{\max}(T_{k-1})}{P^D(t, T_i)(1 - R + S_i \delta_{p_i})}$$

$$Q_{\max}(T_i) = \frac{1 - R - \sum_{k=1}^{i-1} \left((1-R)M_k + S_i N_k\right) Q_{\min}(T_k)}{P^D(t, T_{i-1})(1 - R) + S_i \left(N_i + \delta_{p_i} P^D(t, T_i)\right)}$$

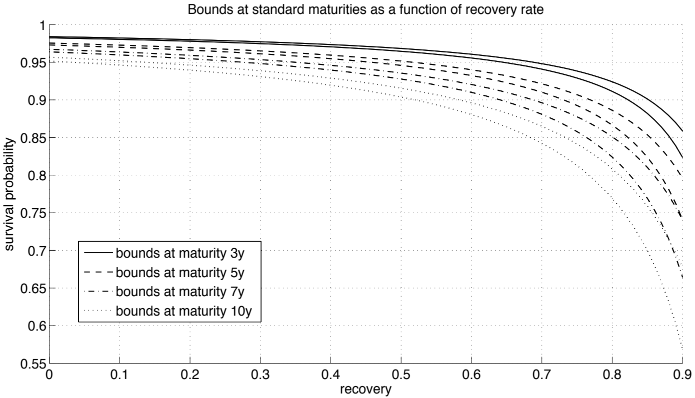
## Range of arbitrage-free market-consistent survival curves




**Input data** : CDS spreads of AIG as of December 17, 2007, $R = 40\%$,
$P^D(t_0, t) = \exp(-3\%(t - t_0))$

# Arbitrage-free bounds for survival curves

**Bounds sensitivity with respect to the recovery rate assumption**




**Input data** : CDS spreads of AIG as of December 17, 2007,

$$P^D(t_0, t) = \exp(-3\%(t - t_0))$$

# How to construct admissible yield curves ?

The yield-curve is built from market quotes of a set of standard products

- $t_0$ : quotation date
- $\mathbf{T} = (T_1, \dots, T_n)$ : set of increasing standard maturities, $T_0 = t_0$
- $\mathbf{S} = (S_1, \dots, S_n)$ : corresponding set of market quotes at $t_0$

We assume that present values can be expressed as linear combination of generic elementary quantities comparable to discount factors :

- $P = P^B$, zero-coupon prices as in ▸ Example 1
- $P = P^D$, discount factors as in ▸ Example 2
- $P = Q$, risk-neutral survival probabilities as in ▸ Example 4

In this presentation, we do not treat the case of elementary quantities comparable to forward rates as in ▸ Example 3.

# How to construct admissible yield curves ?

**Mean-reverting term-structure models as generators of admissible yield curves**

The risk-neutral dynamics of (default-free) interest rates or of default intensities is assumed to follow either

- a OU process driven by a Lévy process

  $$dX_t = a(b(t;\mathbf{p},\mathbf{T},\mathbf{S}) - X_t)dt + \sigma dY_{ct},$$

  where $Y$ is a Lévy process with cumulant function $\kappa$ and parameter set $\mathbf{p}_L$

- or an extended CIR process

  $$dX_t = a(b(t;\mathbf{p},\mathbf{T},\mathbf{S}) - X_t)dt + \sigma\sqrt{X_t}dW_t,$$

  where $W$ is a standard Browian motion

Depending on the context, $\mathbf{p} = (X_0, a, \sigma, c, \mathbf{p}_L)$ will denote the parameter set of the Lévy-OU process and $\mathbf{p} = (X_0, a, \sigma)$ the parameter set of the CIR process

# How to construct admissible yield curves ?

In both cases, $b$ is represented by a step function :

$$b(t; \mathbf{p}, \mathbf{T}, \mathbf{S}) = b_i(\mathbf{p}, \mathbf{T}, \mathbf{S}) \;\text{ for }\; T_{i-1} < t \leqslant T_i, \quad i = 1, \ldots, n$$

The vector $\mathbf{b} = (b_1, \ldots, b_n)$ solves the following pseudo-linear system.

## Market-fit linear conditions

The market-fit condition can be restated as a pseudo-linear system

$$\mathbf{A} \cdot \mathbf{P}(\mathbf{b}) = \mathbf{B}$$

where

- $\mathbf{P}(\mathbf{b}) = (P(t_0, t_k; \mathbf{b}))_{k=1,\ldots,m}$ is the $m \times 1$ vector of elementary quantities that appear in the present value formula of instruments used to build the curve (see Examples 1 to 4).
- $\mathbf{A}$ is a $n \times m$ matrix, $\mathbf{B}$ is a $n \times 1$ matrix
- $\mathbf{A}$ and $\mathbf{B}$ only depend on current market quotes $\mathbf{S}$, on standard maturities $\mathbf{T}$ and on products characteristics.

# How to construct admissible yield curves ?

## Proposition (Discount factors in the Lévy-OU approach)

Let $T_{i-1} < t \leqslant T_i$. In the Lévy-OU model, the current value of the discount factor or of an assimilated quantity with maturity time $t$ is given by

$$P(t_0, t; \mathbf{b}) := \mathbb{E}\left[\exp\left(-\int_{t_0}^{t} X_u du\right)\right] = \exp\left(-I(t_0, t, \mathbf{b})\right)$$

where

$$I(t_0, t, \mathbf{b}) := X_0 \phi(t - t_0) + \sum_{k=1}^{i-1} b_k \left(\xi(t - T_{k-1}) - \xi(t - T_k)\right) + b_i \xi(t - T_{i-1}) + c\psi(t - t_0)$$

and functions $\phi$, $\xi$ and $\psi$ are defined by

$$\begin{aligned} \phi(s) &:= \frac{1}{a}\left(1 - e^{-as}\right) \\ \xi(s) &:= s - \phi(s) \\ \psi(s) &:= -\int_0^s \kappa\left(-\sigma\phi(s - \theta)\right) d\theta \end{aligned} \qquad (1)$$

# How to construct admissible yield curves ?

## Proposition (Discount factors in the CIR approach)

Let $T_{i-1} < t \leqslant T_i$. In the CIR model, the current value of the discount factor or of an assimilated quantity with maturity time $t$ is given by

$$P(t_0, t; \mathbf{b}) := \mathbb{E}\left[\exp\left(-\int_{t_0}^{t} X_u du\right)\right] = \exp\left(-I(t_0, t, \mathbf{b})\right)$$

where

$$I(t_0, t, \mathbf{b}) := X_0 \varphi(t - t_0) + \sum_{k=1}^{i-1} b_k \left(\eta(t - T_{k-1}) - \eta(t - T_k)\right) + b_i \eta(t - T_{i-1})$$

and functions $\varphi$ and $\eta$ are defined by

$$\begin{aligned} \varphi(s) &:= \frac{2(1 - e^{-hs})}{h + a + (h - a)e^{-hs}} \\ \eta(s) &:= 2a\left[\frac{s}{h + a} + \frac{1}{\sigma^2}\log\frac{h + a + (h - a)e^{-hs}}{2h}\right] \end{aligned} \qquad (2)$$

where $h := \sqrt{a^2 + 2\sigma^2}$

# How to construct admissible yield curves ?

**Construction of $(b_1, \ldots, b_n)$ by a bootstrap procedure**

For any $i = 1, \ldots, n$, the present value of the instrument with maturity $T_i$

- only depends on $b_1, \ldots, b_i$
- is a monotonic function with respect to $b_i$

The vector $\mathbf{b} = (b_1, \ldots, b_n)$ satisfies a triangular system of non-linear equations that can be solved recursively :

- Find $b_1$ as the solution of

$$\sum_{j=1}^{p_1} \mathbf{A}_{1j} P(t_0, t_j; b_1) = \mathbf{B}_1$$

- Assume $b_1, \ldots, b_{k-1}$ are known, find $b_k$ as the solution of

$$\sum_{j=1}^{p_k} \mathbf{A}_{kj} P(t_0, t_j; b_1, \ldots, b_k) = \mathbf{B}_k$$

# How to construct admissible yield curves ?

**Proposition (smoothness condition)**

A curve $t \rightarrow P(t_0, t)$ constructed from the previous approach satisfies the smoothness condition : it is of class $\mathcal{C}^1$ and the corresponding forward curve (or default density function) is continuous.

**Proof :** Let $b(\cdot)$ be a deterministic function of time, instantaneous forward rates are such that

- Lévy-driven OU

$$f(t_0, t) = X_0 e^{-a(t-t_0)} + a \int_{t_0}^{t} e^{-a(t-u)} b(u) du - c\kappa(-\sigma\phi(t - t_0))$$

  where $\phi$ is defined by (1)

- extended CIR

$$f^{CIR}(t_0, t) = X_0 \varphi'(t - t_0) + a \int_{t_0}^{t} \varphi'(t - u) b(u) du$$

  where $\varphi'$ is the derivative of $\varphi$ given by (2)

# How to construct admissible yield curves ?

Assume that a curve has been constructed from a Lévy-OU term-structure model with positive parameters $(X_0, a, \sigma, c, \mathbf{p}_L)$ :

$$f(t_0, t) = X_0 e^{-a(t-t_0)} + a \sum_{k=1}^{i-1} b_k \left(\phi(t - T_{k-1}) - \phi(t - T_k)\right) + ab_i \phi(t - T_{i-1}) - c\kappa(-\sigma\phi(t - t_0))$$

for any $T_{i-1} \leq t \leq T_i$, $i = 1, \ldots, n$.

**Proposition (arbitrage-free condition in the Lévy-OU approach)**

Assume that the derivative of the Lévy cumulant $\kappa'$ exists and is strictly monotonic on $(-\infty, 0)$. The curve is arbitrage-free on the time interval $(t_0, T_n)$ if and only if, for any $i = 1, \ldots, n$, $f(t_0, T_i) > 0$ and one of the following condition holds :

- $\frac{\partial f}{\partial t}(t_0, T_{i-1}) \frac{\partial f}{\partial t}(t_0, T_i) \geq 0$
- $\frac{\partial f}{\partial t}(t_0, T_{i-1}) \frac{\partial f}{\partial t}(t_0, T_i) < 0$ and $f(t_0, t_i) > 0$ where $t_i$ is such that $\frac{\partial f}{\partial t}(t_0, t_i) = 0$,

# How to construct admissible yield curves ?

Assume that a curve has been constructed from a extended CIR term-structure model with positive parameters $(X_0, a, \sigma)$ :

$$f^{CIR}(t_0, t) = X_0\varphi'(t-t_0)+a\sum_{k=1}^{i-1} b_k \left(\varphi(t - T_{k-1}) - \varphi(t - T_k)\right)+ab_i\varphi(t-T_{i-1})$$

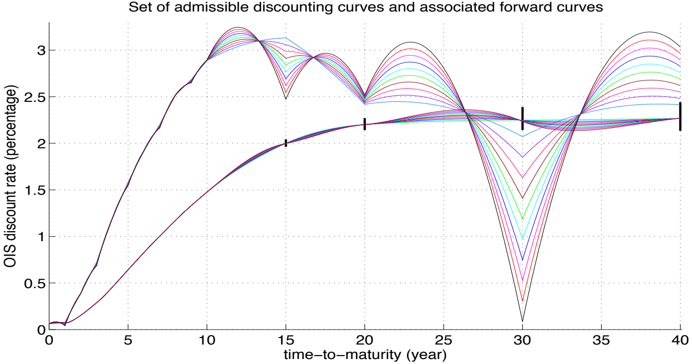
for any $T_{i-1} \leq t \leq T_i$, $i = 1, \ldots, n$.

**Proposition (arbitrage-free condition in the Lévy-OU approach)**

The constructed curve is arbitrage-free if, for any $i = 1, \cdots, n$, the implied $b_i$ is positive

### Set of admissible OIS discount and forward curves : Lévy-OU short rates

**Parameters** : $a = 0.01$, $\sigma = 1$, $X_0 = 0.063\%$ (fair rate of IRS vs OIS 1M). The Lévy driver is a Gamma subordinator with parameter $\lambda = 1/50$bps (mean jump size of 50 bps). $c = \{1, 10, 20, \ldots, 100\}$

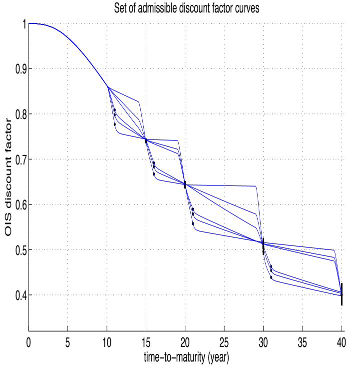


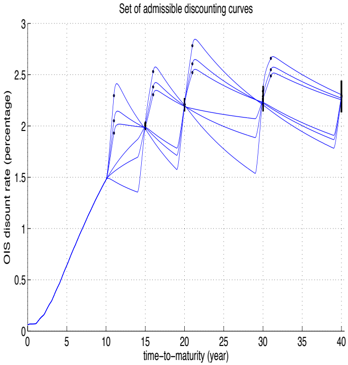

**Input data** : OIS swap rates as of May, 31st 2013

# How to construct admissible yield curves ?

**Arbitrage-free bounds used to generate a wider range of admissible curves**

**Parameters** : CIR short rates with $a = 5$, $\sigma = 1$, $X_0 = 0.063\%$ (fair rate of IRS vs OIS 1M).

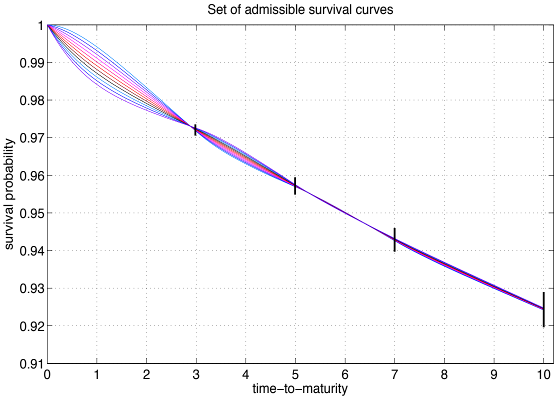



**Input data** : OIS swap rates as of May, 31st 2013

**Set of admissible survival curves : CIR intensities**

**Parameters** : $a = \sigma = 1,\ 100X_0 = \{0.01, 0.25, 0.49,\ 0.73, 0.97,$
$1.21, 1.45, 1.69, 1.94, 2.18, 2.42\}$

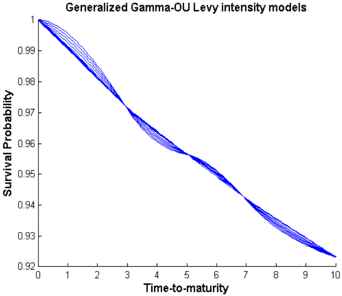


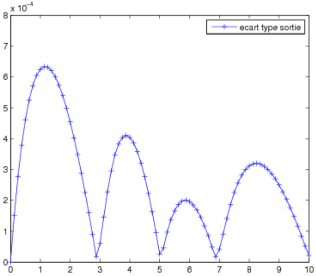

**Input data** : CDS spreads of AIG as of December 17, 2007, $R = 40\%$,
$P^D(t_0, t) = \exp(-3\%(t - t_0))$

# How to construct admissible yield curves ?

**Set of admissible survival curves : Lévy-OU intensities**

**Parameters** : $a$ is Uniform on $[0.5, 10]$, $c$ is Uniform on $[1, 50]$ ($c$ : mean number of jumps per year), $\sigma = 1$. The Lévy driver is a Gamma subordinator with parameter $\lambda = 1/2$bps (mean jump size of 2 bps), $X_0$ is bootstrapped with $b_1$ in such a way that $X_0 = b_1$




**Input data** : CDS spreads of AIG as of December 17, 2007

# Perspectives

The proposed framework could be extended or used in several directions :

- Yield-curve diversity impact on present values (PV) and hedging stategies ?

  $$\max_{i,j} \|PV(C_i) - PV(C_j)\|_p$$

  where the max is taken over all couples of admissible curves $(C_i, C_j)$

- Sensitivity analysis in the presence of uncertain parameters ?

  $$dX_t = \tilde{a}(b(t; \tilde{a}, \tilde{\sigma}, \mathbf{T}, \mathbf{S}) - X_t)dt + \tilde{\sigma}\sqrt{X_t}dW_t,$$

  where $\mathrm{Range}(\tilde{a}, \tilde{\sigma}) \subset \{(a, \sigma) \mid b(t; a, \sigma, \mathbf{T}, \mathbf{S}) \geq 0 \; \forall t\}$

- Extension to a multicurve environment ?

- Impact on the assessment of counterparty credit risk (CVA, EE, EPE, ...) ?

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# Cumulant function of some Lévy processes

| | Cumulant |
|---|---|
| Brownian motion | $\kappa(\theta) = \frac{\theta^2}{2}$ |
| Gamma process | $\kappa(\theta) = -\log\left(1 - \frac{\theta}{\lambda}\right)$ |
| Inverse Gaussian process | $\kappa(\theta) = \lambda - \sqrt{\lambda^2 - 2\theta}$ |