Delta-Hedging Correlation Risk?

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Introduction

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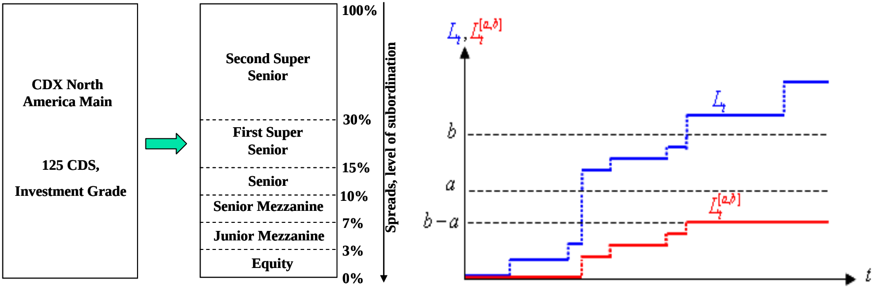
Introduction

- Performance analysis of alternative hedging strategies developed for the correlation market
- CDO tranches on standard Index such as CDX North America Investment Grade index



Introduction

- Credit crisis has deeply affected the CDS index market
  - Series 10 of CDX North America index suffers defaults of Fannie Mae, Freddie Mac and Lehman Brothers
  - High level of credit spreads and volatility
- Recent revision of Basel II regulation concerns risk-management of credit derivatives
  - Residual risks resulting from dynamic hedging strategies must be reflected in the capital charge
- Performance and efficiency of underlying hedging methods is a topical issue

Hedging loss derivatives

Generally speaking, ...

Hedging derivative instruments consists in taking opposite positions in some primary liquid instruments whose market values are sensitive to the same underlying risks

- The aim is to minimize the overall exposure to market price evolution
- Composition of the hedging portfolio need to be regularly updated over time
- Require a pricing device based on a model of portfolio credit risk
- Daily recalibration of model parameters on market quotes

Hedging loss derivatives

In this study, ...

- Hedging of a buy protection position on an index CDO tranche
- Hedging portfolio composed of two instruments:
    - CDS Index
    - Savings account

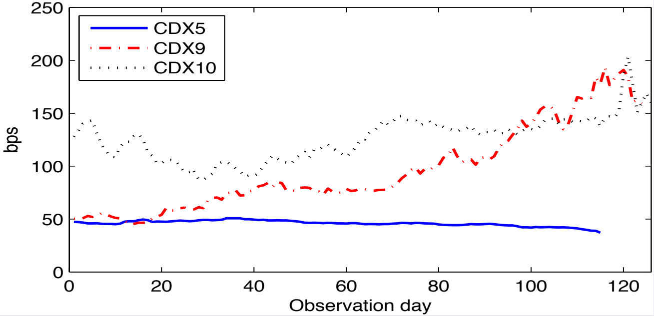
Performance analysis of alternative hedging methods:

- Δ^{Gauss}: delta of the tranche computed within the one-factor Gaussian copula model (standard quotation device)
- Δ^{lo}: delta of the tranche computed within the local intensity model (two specifications of model parameters)

Data set

- 5-year CDX NA IG Series 5 from 20 September 2005 to 20 March 2006
- 5-year CDX NA IG Series 9 from 20 September 2007 to 20 March 2008
- 5-year CDX NA IG Series 10 from 21 March 2008 to 20 September 2008

Index spreads



Notations

- Credit portfolio with n reference entities
- $\tau_1, \ldots, \tau_n$: default times
- R: homogeneous and constant recovery rate at default (In numerical investigations, we consider $R = 40\%$)
- Number of defaults process:

$$N_t = \sum_{i=1}^{n} 1_{\{\tau_i \leq t\}}$$

- CDO tranche cash-flows are driven by the aggregate loss process normalized to unity:

$$L_t = \frac{1}{n}(1 - R)N_t$$

- CDO tranche cash-flows only depend on $\varphi(L_t)$, $0 \leq t \leq T$

One factor Gaussian copula model

- $V_i = \rho V + \sqrt{1-\rho^2}\bar{V}_i,\ i = 1 \dots n$: latent variables
  - $V, \bar{V}_i,\ i = 1 \dots n$: independent Gaussian random variables
- Default times defined by: $\tau_i = F_i^{-1}(\Phi(V_i)),\ i = 1 \dots n$
  - $F_1 = \dots = F_n = F$: cdf of τ_i, $i = 1, \dots, n$
  - Φ: cdf of V_i
- Conditional default probability

$$p_t(V) = \mathbb{P}(\tau_i \le t \mid V) = \Phi\left(\frac{\Phi^{-1}\left(F(t)\right) - \rho V}{\sqrt{1-\rho^2}}\right)$$

- Loss distribution is merely a binomial mixture:

$$\mathbb{P}(N_t = k) = \begin{pmatrix} n \\ k \end{pmatrix} \int p_t(x)^k (1 - p_t(x))^{n-k} \nu(x) dx, \quad k = 0, \dots, n$$

One factor Gaussian copula model

- At each time t, the model parameters ρ_t and F_t are calibrated on market spreads
- F_t is inferred from the term structure of index spreads at time t
  - Index spread curve assumed to be flat and equal to S_t (5-years spread)

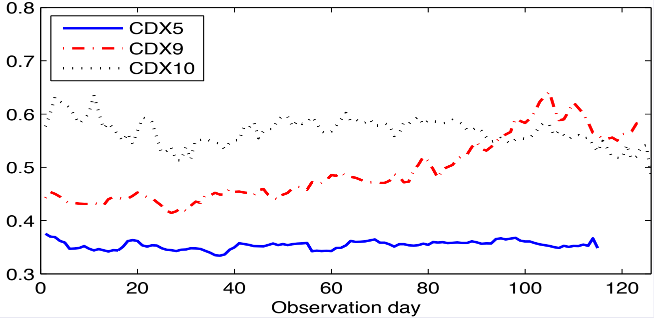
$$F_t(x) = \mathbb{P}(\tau_i \leq x) = 1 - \exp\left(-\frac{S_t}{1-R}x\right)$$

- One dependence parameter ρ_t^b associated with each base tranche $[0, b]$, $b =$ 3%, 7%, 10%, 15%, 30% (CDX)
  - $\Pi_t^{ma}(T, a, b)$: market price of CDO tranche $[a, b]$, maturity T
  - $\Pi^{gc}(T, a, b;\, t, S_t, \rho_t)$: price of CDO tranche $[a, b]$ in the Gaussian copula model
  - Base correlation ρ_t^b is such that:

$$\Pi^{gc}(T, 0, b;\, t, S_t, \rho_t^b) = \Pi_t^{ma}(T, 0, b)$$

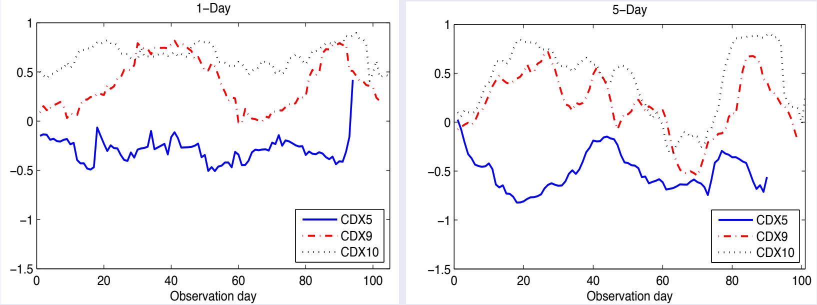
One factor Gaussian copula model

Base correlation at 3% strike



One factor Gaussian copula model

Correlation between index spread return and 3% base correlation return based on 20-day rolling window. Left: 1-day returns; Right: 5-day returns.



Local intensity model

- Parallels the Dupire's local volatility approach developed for the equity derivative market
- The number of defaults N_t is modeled as a continuous-time Markov chain (pure birth process) with generator matrix:

$$\Lambda(t) = \begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & & 0 \\ & & \ddots & \ddots & \\ 0 & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- $\lambda(t,k)$, $k = 0,\dots,n-1$: state-dependent default intensities
- The time-T loss distribution $P(N_T = k \mid N_t)$ satisfies a $n+1$-dimensional system of backward Kolmogorov equations that can be solved numerically

Hedging loss derivatives

Gauss delta:

$$\Delta_t^{\text{Gauss}} = \frac{\Pi^{gc}(T,a,b;\, t, S_t + \varepsilon, \rho_t) - \Pi^{gc}(T,a,b;\, t, S_t, \rho_t)}{\Pi^{gc}(T,0,1;\, t, S_t + \varepsilon) - \Pi^{gc}(T,0,1;\, t, S_t)}$$

- $\Pi^{gc}(T,a,b;\, .)$: price of T-year protection tranche [a,b] computed in the Gaussian copula model
- $\Pi^{gc}(T,0,1;\, .)$: price of the T-year CDX index computed in the Gaussian copula model
- S_t: credit spread of the CDS index at time t
- $\varepsilon = 1$ bp
- ρ_t: implied correlation parameter of the tranche at time t. Sticky-strike rule: base correlation is kept unchanged when bumping the index credit curve

Gauss delta = Sensitivity with respect to the CDS Index spread using the industry standard quotation device

Hedging loss derivatives

Local intensity delta:

$$\Delta_t^{\mathrm{lo}} = \frac{\Pi^{lo}\left(T, a, b;\, t, N_t + 1\right) - \Pi^{lo}\left(T, a, b;\, t, N_t\right)}{\Pi^{lo}\left(T, 0, 1;\, t, N_t + 1\right) - \Pi^{lo}\left(T, 0, 1;\, t, N_t\right)}.$$

- $\Pi^{lo}\left(T, a, b;\, .\right)$: price of the tranche computed in the local intensity model
- $\Pi^{lo}\left(T, 0, 1;\, .\right)$: price of the CDX index computed in the local intensity model
- N_t: current number of defaults

Local intensity delta = Jump-to-Default delta computed using the local intensity model

Model Specifications

- **Gauss**: Gaussian copula model with one implied correlation parameter per standard tranche (base correlation approach)
- **Para**: Local intensity model – parametric specification of local itensities

$$\lambda(t,k) = \lambda(k) = (n-k)\sum_{i=0}^{k} b_i$$

(Herbertsson (2008))

- **EM**: Local intensity model – local itensities $\lambda(t,k)$ obtained by minimizing a relative entropy distance with respect to a prior distribution

$$\inf_{\mathbb{Q}\in\Lambda} \mathbb{E}^{\mathbb{Q}_0}\left[\frac{d\mathbb{Q}}{d\mathbb{Q}_0}\ln\left(\frac{d\mathbb{Q}}{d\mathbb{Q}_0}\right)\right]$$

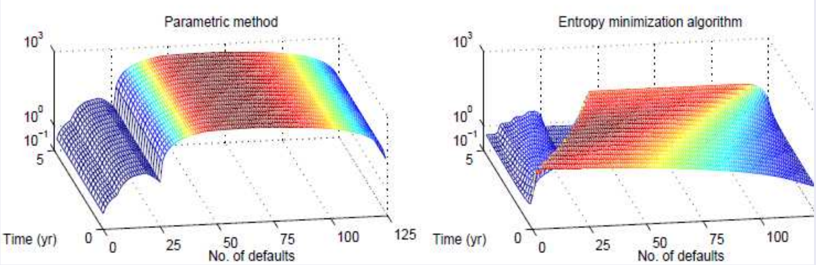
(Cont and Minca (2008))

Empirical results

Root mean squared calibration errors (in percentage):

	CDX5			CDX9			CDX10		
Tranche	Gauss	Para	EM	Gauss	Para	EM	Gauss	Para	EM
Index	0.04	5.15	5.14	0.03	4.40	4.81	0.02	6.73	6.77
0%-3%	0.01	2.35	2.36	0.00	1.31	1.32	0.01	1.69	1.68
3%-7%	0.00	0.51	0.69	0.00	0.61	0.86	0.00	1.04	1.03
7%-10%	0.00	0.08	1.32	0.00	0.24	0.91	0.00	0.43	0.39
10%-15%	0.00	0.06	1.77	0.00	0.24	1.15	0.00	0.40	0.36
15%-30%	0.00	0.29	1.97	0.01	1.19	1.74	0.01	1.80	1.68

Comparison of typical shapes of local intensities $\lambda(t, k)$, Para (left), EM (right)



Calibration results

Comparison of three alternative hedging methods

- Gauss delta: index Spread sensitivity computed in a one-factor Gaussian copula model

$$\Delta_t^{\text{Gauss}} = \frac{\Pi^{gc}(T,a,b;\,t,S_t+\varepsilon,\rho_t) - \Pi^{gc}(T,a,b;\,t,S_t,\rho_t)}{\Pi^{gc}(T,0,1;\,t,S_t+\varepsilon) - \Pi^{gc}(T,0,1;\,t,S_t)}$$

where $\mathcal{V}$ and $\mathcal{V}^I$ are the Gaussian copula pricing function associated with (resp.) the tranche and the CDS index.

- Local intensity delta:

$$\Delta_t^{\text{lo}} = \frac{\Pi^{lo}(T,a,b;\,t,N_t+1) - \Pi^{lo}(T,a,b;\,t,N_t)}{\Pi^{lo}(T,0,1;\,t,N_t+1) - \Pi^{lo}(T,0,1;\,t,N_t)}.$$

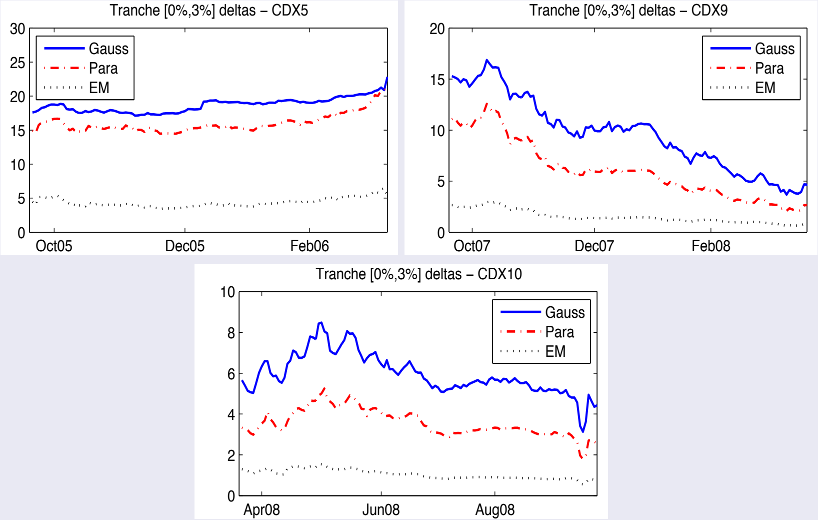
with both Parametric (Para) and Entropy Minimisation (EM) calibration methods

Credit deltas on 20 September 2007 (normalized to tranche notional)

Tranche	Gauss	Para	EM
0%-3%	15.29	11.05	2.64
3%-7%	5.03	4.59	2.70
7%-10%	1.94	2.26	2.29
10%-15%	1.10	1.47	1.99
15%-30%	0.60	1.01	1.74

Empirical results

Time series of equity tranche credit deltas, CDX.NA.IG series 5, 9 and 10



Hedging performance

Back-testing hedging experiments on series 5, 9 and 10

- Hedging portfolio rebalanced everyday (dt=1) or every 5 days (dt=5)
- P&L (Profit-and-Loss) increment of hedged position:

$$\delta P\&L(t) = \delta\Pi_m(t) - \Delta_t \cdot \delta\Pi_m^I(t)$$

  - $\delta\Pi_m(t) = \Pi_m(t+dt) - \Pi_m(t)$: Increment of tranche market value
  - $\delta\Pi_m^I(t) = \Pi_m^I(t+dt) - \Pi_m^I(t)$: Increment of index market value
  - Δ_t: One of the previous hedging ratios computed at time t
- P&L increments evaluated in the same frequency as rebalancing

Hedging performance

Two metrics to compare the hedging strategies:

$$\textbf{Relative hedging error} = \left| \frac{\text{Average P\&L increment of the hedged position}}{\text{Average P\&L increment of the unhedged position}} \right| = \left| \frac{\text{Average of } \delta P\&L(t)}{\text{Average of } \delta V_m(t)} \right|$$

$$\textbf{Residual volatility} = \frac{\text{P\&L increment volatility of the hedged position}}{\text{P\&L increment volatility of the unhedged position}} = \frac{\text{Volatility of } \delta P\&L(t)}{\text{Volatility of } \delta V_m(t)}$$

Hedging performance for 1-day rebalancing

Relative hedging errors (in percentage)

	CDX5			CDX9			CDX10		
Tranche	Li	Para	EM	Li	Para	EM	Li	Para	EM
0%-3%	4	5	73	80	10	72	33	55	90
3%-7%	1	3	35	0.4	19	59	48	49	75
7%-10%	10	10	43	15	13	37	49	25	44
10%-15%	7	27	131	27	18	14	139	181	208
15%-30%	0.54	61	324	3	32	89	172	269	396

Residual volatilities (in percentage)

	CDX5			CDX9			CDX10		
Tranche	Gauss	Para	EM	Gauss	Para	EM	Gauss	Para	EM
0%-3%	45	47	79	59	59	87	105	91	93
3%-7%	70	72	68	58	47	64	85	74	78
7%-10%	90	101	120	53	50	46	83	79	70
10%-15%	90	107	188	61	63	60	91	93	86
15%-30%	93	110	256	37	49	77	84	99	127

Hedging performance for 5-days rebalancing

Relative hedging errors (in percentage)

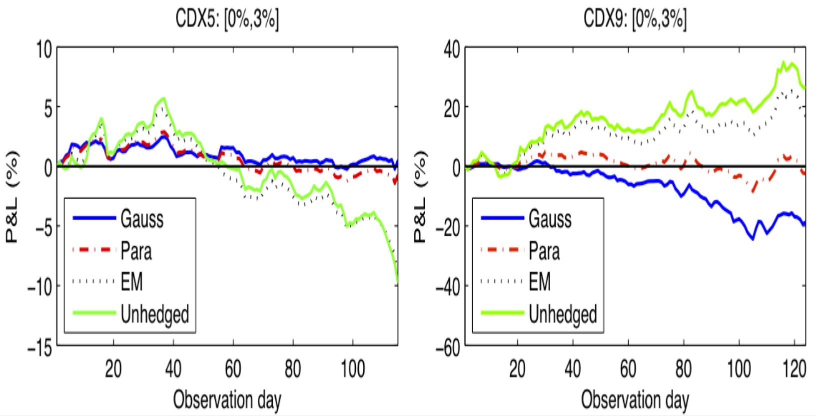
	CDX5			CDX9			CDX10		
Tranche	Gauss	Para	EM	Gauss	Para	EM	Gauss	Para	EM
0%-3%	6	10	77	59	2	73	24	48	88
3%-7%	16	16	51	2	18	58	48	43	72
7%-10%	19	1	15	11	12	36	50	15	41
10%-15%	22	8	75	13	5	5	141	198	209
15%-30%	21	30	207	1	35	86	127	227	382

Residual volatilities (in percentage)

	CDX5			CDX9			CDX10		
Tranche	Gauss	Para	EM	Gauss	Para	EM	Gauss	Para	EM
0%-3%	42	46	83	50	56	86	71	72	89
3%-7%	75	75	66	73	65	71	43	40	64
7%-10%	99	118	135	57	56	54	40	38	44
10%-15%	82	110	202	94	98	95	42	44	40
15%-30%	77	108	298	46	69	108	31	33	54

Hedging performance for 5-days rebalancing

Path of cumulative P&L of hedged and unhedged positions in equity tranche



Conclusion

- All model specifications perfectly fit CDO tranche quotes
- However, for the local intensity model, the two introduced specifications give strikingly different deltas and dramatically different hedging performances
- Hedging based on local intensity model with Entropy Minimisation calibration gives poor performance
- Performance of hedging based on the Gaussian copula model and on the parametric local intensity model are comparable for crisis period associated with CDX Series 9 and 10.
- However, the local intensity delta fails to outperform the market delta in pre-crisis period associated with CDX Series 5,

Thank you for your attention!

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