Model Risk Embedded in Yield-Curve Construction Methods

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Introduction

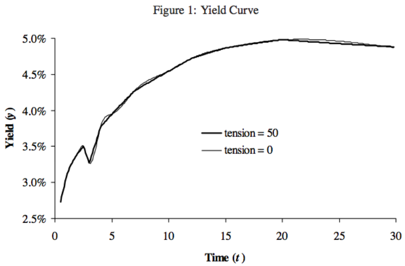
Cousin, A. and Niang, I.

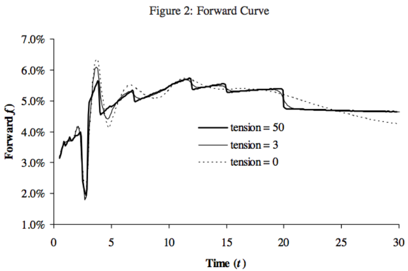
On the range of admissible term-structures

Available on arXiv.org : http ://arxiv.org/abs/1404.0340

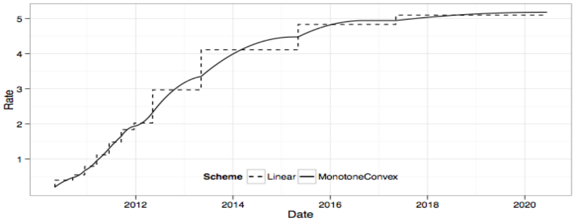
Introduction

Andersen (2007), curves based on tension splines





Le Floc'h (2012), examples of one-day forward curves



Introduction

Admissible curve

A yield curve is said to be admissible if it satisfies the following constraints :

- The curve is market-consistent : it perfectly reproduces input data
- The curve is arbitrage-free : forward rates are positive
- The curve is smooth enough : forward rates are at least continuous

We then address the following questions :

- Is it possible to estimate the size of admissible curves ? and how ?
- How does the range/diversity of admissible curves affect the present value of products with non-standard characteristics ?

Instruments used for curve construction

Assumption : Linear representation of present values

Presents values of products used in the curve construction can be expressed linearly with respect to some elementary quantities such as zero-coupon prices or discount factors

Example 1 : Corporate or sovereign debt yield curve

- S : market price (in percentage of nominal) at time t_0 of a bond with maturity T
- c : fixed coupon rate
- $t_1 < \ldots < t_p = T$: coupon payment dates, δ_k : year fraction corresponding to period (t_{k-1}, t_k)

$$c \sum_{k=1}^{p} \delta_k P^B(t_0, t_k) + P^B(t_0, T) = S$$

where $P^B(t_0, t_k)$ represents the price of a (fictitious default-free issuer-dependent) ZC bond with maturity t_k

Example 2 : Discounting curve based on OIS

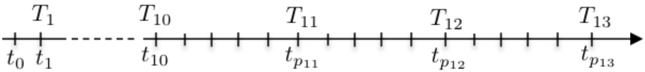
- S^{OIS} : par rate at time t_0 of an overnight indexed swap with maturity T
- $t_1 < \cdots < t_p = T$: fixed-leg payment dates
- δ_k : year fraction corresponding to period (t_{k-1}, t_k)

$$S^{\text{OIS}} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) = 1 - P^D(t_0, T)$$

where $P^D(t_0, t_k)$ is the discount factor associated with maturity date t_k

Arbitrage-free bounds for OIS discount curves

- We observe OIS par rates $S_1, \cdots, S_n$ for maturities $T_1 < \cdots < T_n$.
- Let $t = t_0 < t_1 < \cdots < t_{p_n} = T_n$ be the annual time grid up to time T_n.
- The set of indices (p_i) is such that $t_{p_i} = T_i$ for $i = 1, ..., n$.



Arbitrage-free bounds for OIS discount curves

- Market-consistency condition translates into a rectangular system of linear constraints :

$$S_i \sum_{k=1}^{p_i - 1} \delta_k P^D(t_0, t_k) + (S_i \delta_{p_i} + 1) P^D(t_0, T_i) = 1, \quad i = 1, ..., n$$

- Let i_0 be the smallest index such that $T_{i_0} \neq t_{i_0}$ ($i_0 = 11$ in our applications)

- Define $H_i := \sum_{k=p_{i-1}+1}^{p_i - 1} \delta_k$, for $i = i_0, \ldots, n$

Arbitrage-free bounds for OIS discount curves

Proposition (arbitrage-free bounds for discount factors)

$$P^D(t_0, T_1) = \frac{1}{1 + S_1\delta_1},$$

$$P^D(t_0, T_i) = \frac{1}{1 + S_i\delta_i}\left(1 - \frac{S_i}{S_{i-1}}\left(1 - P^D(t_0, T_{i-1})\right)\right), \quad i = 2, \ldots, i_0 - 1$$

For $i = i_0, \ldots, n$,

$$P^D_{\min}(t_0, T_i) \leqslant P^D(t_0, T_i) \leqslant P^D_{\max}(t_0, T_i)$$

where

$$P^D_{\min}(t_0, T_i) = \frac{1}{1 + S_i\delta_{p_i}}\left(1 - \frac{S_i}{S_{i-1}}\left(1 - (1 - S_{i-1}H_i)P^D(t_0, T_{i-1})\right)\right)$$

$$P^D_{\max}(t_0, T_i) = \frac{1}{1 + S_i(H_i + \delta_{p_i})}\left(1 - \frac{S_i}{S_{i-1}}\left(1 - P^D(t_0, T_{i-1})\right)\right)$$

Iterative computation of model-free bounds

- **Step 1** : For $i = 1, \ldots, i_0 - 1$,

$$P^D(t_0, T_i) = \frac{1}{1 + S_i \delta_i} \left(1 - \frac{S_i}{S_{i-1}} \left(1 - P^D(t_0, T_{i-1})\right)\right)$$

- **Step 2** : For $i = i_0, \ldots, n$,

$$P_{\min}(T_i) \leqslant P^D(t_0, T_i) \leqslant P_{\max}(T_i)$$

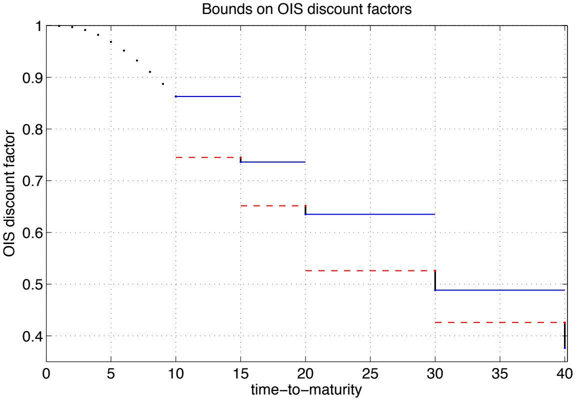
where

$$P_{\min}(T_i) = \frac{1}{1 + S_i \delta_{p_i}} \left(1 - \frac{S_i}{S_{i-1}} (1 - (1 - S_{i-1} H_i) P_{\min}(T_{i-1}))\right)$$

$$P_{\max}(T_i) = \frac{1}{1 + S_i (H_i + \delta_{p_i})} \left(1 - \frac{S_i}{S_{i-1}} (1 - P_{\max}(T_{i-1}))\right)$$

Arbitrage-free bounds for OIS discount curves

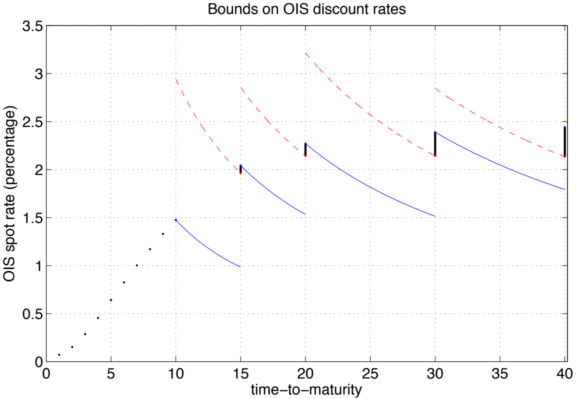
The previous bounds are sharp



Input data : OIS swap rates as of May, 31st 2013

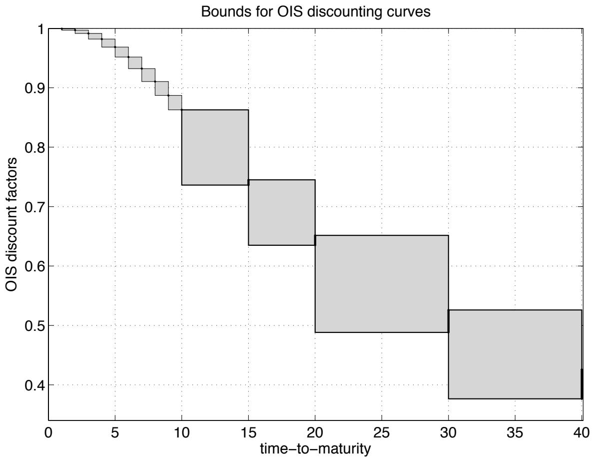
Arbitrage-free bounds for OIS discount curves

Model-free bounds for the associated discount rates



Input data : OIS swap rates as of May, 31st 2013, $-\frac{1}{T}\log(P^D(t_0, T))$

Range of arbitrage-free market-consistent OIS discount curves



Input data : OIS swap rates as of May, 31st 2013

How to construct admissible yield curves ?

The yield-curve is built from market quotes of a set of standard products

- t_0 : quotation date
- $\mathbf{T} = (T_1, \ldots, T_n)$: set of increasing standard maturities, $T_0 = t_0$
- $\mathbf{S} = (S_1, \ldots, S_n)$: corresponding set of market quotes at t_0

We assume that present values can be expressed as linear combinations of zero-coupon prices or discount factors :

- $P = P^B$, zero-coupon prices as in ▸ Example 1
- $P = P^D$, discount factors as in ▸ Example 2

How to construct admissible yield curves ?

Mean-reverting term-structure models as generators of admissible yield curves

The risk-neutral dynamics of (default-free) interest rates is assumed to follow either

- a OU process driven by a Lévy process

$$dX_t = a(b(t; \mathbf{p}, \mathbf{T}, \mathbf{S}) - X_t)dt + \sigma dY_{ct},$$

  where Y is a Lévy process with cumulant function κ and parameter set $\mathbf{p}_L$

- or an extended CIR process

$$dX_t = a(b(t; \mathbf{p}, \mathbf{T}, \mathbf{S}) - X_t)dt + \sigma\sqrt{X_t}dW_t,$$

  where W is a standard Browian motion

Depending on the context, $\mathbf{p} = (X_0, a, \sigma, c, \mathbf{p}_L)$ will denote the parameter set of the Lévy-OU process and $\mathbf{p} = (X_0, a, \sigma)$ the parameter set of the CIR process

How to construct admissible yield curves ?

In both cases, b is represented by a step function :

$$b(t; \mathbf{p}, \mathbf{T}, \mathbf{S}) = b_i(\mathbf{p}, \mathbf{T}, \mathbf{S}) \text{ for } T_{i-1} < t \leqslant T_i, \quad i = 1, \dots, n$$

The vector $\mathbf{b} = (b_1, \dots, b_n)$ solves the following pseudo-linear system.

Market-fit linear conditions

The market-fit condition can be restated as a pseudo-linear system

$$\mathbf{A} \cdot \mathbf{P}(\mathbf{b}) = \mathbf{B}$$

where

- $\mathbf{P}(\mathbf{b}) = (P(t_0, t_k; \mathbf{b}))_{k=1,\dots,m}$ is the $m \times 1$ vector of elementary quantities that appear in the present value formula of instruments used to build the curve (see Examples 1 and 2).
- $\mathbf{A}$ is a $n \times m$ matrix, $\mathbf{B}$ is a $n \times 1$ matrix
- $\mathbf{A}$ and $\mathbf{B}$ only depend on current market quotes $\mathbf{S}$, on standard maturities $\mathbf{T}$ and on products characteristics.

How to construct admissible yield curves ?

Proposition (Discount factors in the Lévy-OU approach)

Let $T_{i-1} < t \leqslant T_i$. In the Lévy-OU model, the current value of the discount factor or of an assimilated quantity with maturity time t is given by

$$P(t_0, t; \mathbf{b}) := \mathbb{E}\left[\exp\left(-\int_{t_0}^{t} X_u du\right)\right] = \exp\left(-I(t_0, t, \mathbf{b})\right)$$

where

$$\begin{aligned} I(t_0, t, \mathbf{b}) := X_0 \phi(t - t_0) &+ \sum_{k=1}^{i-1} b_k \left(\xi(t - T_{k-1}) - \xi(t - T_k)\right) \\ &+ b_i \xi(t - T_{i-1}) + c\psi(t - t_0) \end{aligned}$$

and functions ϕ, ξ and ψ are defined by

$$\begin{aligned} \phi(s) &:= \frac{1}{a}\left(1 - e^{-as}\right) \\ \xi(s) &:= s - \phi(s) \\ \psi(s) &:= -\int_0^s \kappa\left(-\sigma\phi(s - \theta)\right) d\theta \end{aligned} \tag{1}$$

How to construct admissible yield curves ?

Proposition (Discount factors in the CIR approach)

Let $T_{i-1} < t \leqslant T_i$. In the CIR model, the current value of the discount factor or of an assimilated quantity with maturity time t is given by

$$P(t_0, t; \mathbf{b}) := \mathbb{E}\left[\exp\left(-\int_{t_0}^{t} X_u du\right)\right] = \exp\left(-I(t_0, t, \mathbf{b})\right)$$

where

$$I(t_0, t, \mathbf{b}) := X_0 \varphi(t - t_0) + \sum_{k=1}^{i-1} b_k \left(\eta(t - T_{k-1}) - \eta(t - T_k)\right) + b_i \eta(t - T_{i-1})$$

and functions φ and η are defined by

$$\begin{aligned} \varphi(s) &:= \frac{2(1 - e^{-hs})}{h + a + (h - a)e^{-hs}} \\ \eta(s) &:= 2a\left[\frac{s}{h+a} + \frac{1}{\sigma^2}\log\frac{h + a + (h - a)e^{-hs}}{2h}\right] \end{aligned} \quad (2)$$

where $h := \sqrt{a^2 + 2\sigma^2}$

How to construct admissible yield curves ?

Construction of $(b_1, \dots, b_n)$ by a bootstrap procedure

For any $i = 1, \dots, n$, the present value of the instrument with maturity T_i

- only depends on $b_1, \dots, b_i$
- is a monotonic function with respect to b_i

The vector $\mathbf{b} = (b_1, \dots, b_n)$ satisfies a triangular system of non-linear equations that can be solved recursively :

- Find b_1 as the solution of

$$\sum_{j=1}^{p_1} \mathbf{A}_{1j} P(t_0, t_j; b_1) = \mathbf{B}_1$$

- Assume $b_1, \dots, b_{k-1}$ are known, find b_k as the solution of

$$\sum_{j=1}^{p_k} \mathbf{A}_{kj} P(t_0, t_j; b_1, \dots, b_k) = \mathbf{B}_k$$

How to construct admissible yield curves ?

Proposition (smoothness condition)

A curve $t \to P(t_0, t)$ constructed from the previous approach satisfies the smoothness condition : it is of class $\mathcal{C}^1$ and the corresponding forward curve (or default density function) is continuous.

Proof : Let $b(\cdot)$ be a deterministic function of time, instantaneous forward rates are such that

- Lévy-driven OU

$$f(t_0, t) = X_0 e^{-a(t-t_0)} + a \int_{t_0}^{t} e^{-a(t-u)} b(u) du - c\kappa(-\sigma\phi(t - t_0))$$

  where ϕ is defined by (1)

- extended CIR

$$f^{CIR}(t_0, t) = X_0 \varphi'(t - t_0) + a \int_{t_0}^{t} \varphi'(t - u) b(u) du$$

  where φ' is the derivative of φ given by (2)

How to construct admissible yield curves ?

Assume that a curve has been constructed from a Lévy-OU term-structure model with positive parameters $(X_0, a, \sigma, c, \mathbf{p}_L)$:

$$f(t_0, t) = X_0 e^{-a(t-t_0)} + a \sum_{k=1}^{i-1} b_k \left(\phi(t - T_{k-1}) - \phi(t - T_k)\right) + a b_i \phi(t - T_{i-1}) - c\kappa(-\sigma\phi(t - t_0))$$

for any $T_{i-1} \leq t \leq T_i$, $i = 1, \ldots, n$.

Proposition (arbitrage-free condition in the Lévy-OU approach)

Assume that the derivative of the Lévy cumulant κ' exists and is strictly monotonic on $(-\infty, 0)$. The curve is arbitrage-free on the time interval (t_0, T_n) if and only if, for any $i = 1, \ldots, n$, $f(t_0, T_i) > 0$ and one of the following condition holds :

- $\dfrac{\partial f}{\partial t}(t_0, T_{i-1}) \dfrac{\partial f}{\partial t}(t_0, T_i) \geq 0$
- $\dfrac{\partial f}{\partial t}(t_0, T_{i-1}) \dfrac{\partial f}{\partial t}(t_0, T_i) < 0$ and $f(t_0, t_i) > 0$ where t_i is such that $\dfrac{\partial f}{\partial t}(t_0, t_i) = 0$,

How to construct admissible yield curves ?

Assume that a curve has been constructed from an extended CIR term-structure model with positive parameters (X_0, a, σ) :

$$f^{CIR}(t_0, t) = X_0\varphi'(t-t_0)+a\sum_{k=1}^{i-1} b_k\left(\varphi(t-T_{k-1})-\varphi(t-T_k)\right)+ab_i\varphi(t-T_{i-1})$$

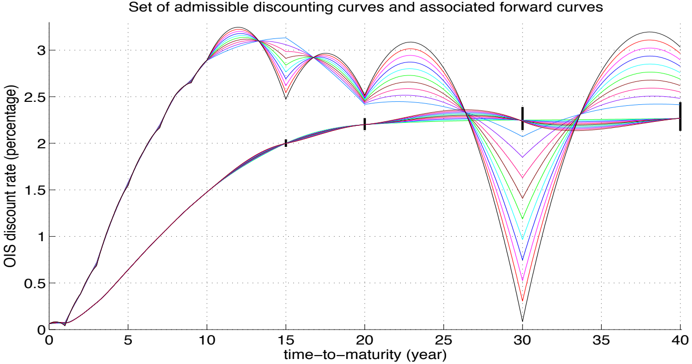
for any $T_{i-1} \leq t \leq T_i$, $i = 1, \ldots, n$.

Proposition (arbitrage-free condition in the CIR approach)

The constructed curve is arbitrage-free if, for any $i = 1, \cdots, n$, the implied b_i is positive

Set of admissible OIS discount and forward curves : Lévy-OU short rates

Parameters : $a = 0.01$, $\sigma = 1$, $X_0 = 0.063\%$ (fair rate of IRS vs OIS 1M). The Lévy driver is a Gamma subordinator with parameter $\lambda = 1/50$bps (mean jump size of 50 bps). $c = \{1, 10, 20, \ldots, 100\}$



Input data : OIS swap rates as of May, 31st 2013

Perspectives

The proposed framework could be extended or used in several directions :

- Yield-curve diversity impact on present values (PV) and hedging stategies ?

$$\max_{i,j} \|PV(C_i) - PV(C_j)\|_p$$

  where the max is taken over all couples of admissible curves (C_i, C_j)

- Sensitivity analysis in the presence of uncertain parameters ?

$$dX_t = \tilde{a}(b(t; \tilde{a}, \tilde{\sigma}, \mathbf{T}, \mathbf{S}) - X_t)dt + \tilde{\sigma}\sqrt{X_t}dW_t,$$

  where $\mathrm{Range}(\tilde{a}, \tilde{\sigma}) \subset \{(a, \sigma) \mid b(t; a, \sigma, \mathbf{T}, \mathbf{S}) \geq 0 \ \forall t\}$

- Extension to a multicurve environment ?

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Cumulant function of some Lévy processes

	Cumulant
Brownian motion	$\kappa(\theta) = \frac{\theta^2}{2}$
Gamma process	$\kappa(\theta) = -\log\left(1 - \frac{\theta}{\lambda}\right)$
Inverse Gaussian process	$\kappa(\theta) = \lambda - \sqrt{\lambda^2 - 2\theta}$