Model risk embedded in yield curve construction methods

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Introduction

- What is understood as a term-structure in this presentation?
- What is it used for?
Term-structures are constructed from market quotes of fixed income, fx or default-risky securities.

Information provided by the market is reliable only for a small set of liquid products with standard characteristics/maturities.

We have to rely on interpolation/calibration schemes to construct the curve for missing maturities.

A variety of curve construction methods exists: no consensus towards a particular best practice in all circumstances.
Andersen (2007), curves based on tension splines

Le Floc’h (2012), examples of one-day forward curves
What can be defined as a good yield curve construction method? (Hagan and West (2006))

- Ability to fit market quotes
- Arbitrage freeness
- Smoothness
- Locality of the interpolation method
- Stability of forward rate
- Consistency of hedging strategies: Locality of deltas? Sum of sequential deltas close enough to the corresponding parallel delta? (Le Floc’h (2012))
Interestingly, there is a pretty large recent literature on the subject of yield-curve construction methods.


And a flourishing literature on model risk

Arbitrage-free curve

A curve is said to be arbitrage-free if

- **IR curves**: the forward rates are non-negative or equivalently, the (pseudo) discount factors are nonincreasing with respect to time-to-maturities.
- **Credit**: the curve is associated with a well-defined default distribution function.

Smoothness condition

A curve is said to be smooth if

- **IR curves**: the instantaneous forward rates exist for all maturities and are continuous.
- **Credit**: the default density function exists and is continuous.
Admissible curve

A yield curve is said to be admissible if it satisfies the following constraints:

- The input data set is perfectly reproduced by the curve
- The curve is arbitrage-free
- The curve satisfies the smoothness condition
We then address the following questions:

- Is it possible to estimate the size of admissible curves? and how?
- How does the range/diversity of admissible curves affect the present value of products with non-standard characteristics?

We develop a framework in which it is possible to measure the diversity of yield curves with some specific features.
Instruments used for curve construction

Assumption I: pseudo-linear representation of present values

Products used in the curve construction have presents values that can be expressed as linear combination of some elementary quantities such as zero-coupon prices, discount factors, Ibor forward rates or survival probabilities.

Example 1: Corporate or sovereign debt yield curve

- $S$: market price (in percentage of nominal) at time $t_0$ of a bond with maturity $T$
- $c$: fixed coupon rate
- $t_1 < \ldots < t_p = T$: coupon payment dates, $\delta_k$: year fraction corresponding to period $(t_{k-1}, t_k)$

\[
c \sum_{k=1}^{p} \delta_k P^B(t_0, t_k) + P^B(t_0, T) = S
\]

where $P^B(t_0, t_k)$ represents the price of a (fictitious default-free issuer-dependent) ZC bond with maturity $t_k$
Example 2: Discounting curve based on OIS

- \( S^{\text{OIS}} \): par rate at time \( t_0 \) of an overnight indexed swap with maturity \( T \)
- \( t_1 < \cdots < t_p = T \): fixed-leg payment dates
- \( \delta_k \): year fraction corresponding to period \((t_{k-1}, t_k)\)

\[
S^{\text{OIS}} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) = 1 - P^D(t_0, T)
\]

where \( P^D(t_0, t_k) \) is the discount factor associated with maturity date \( t_k \)
Example 3: Forward curve based on fixed-vs-Ibor-floating IRS

- $S_{\text{IRS}}$: par rate at time $t_0$ of an interest rate swap with maturity $T$ and tenor $j$ (typically, $j = 3M$ or $j = 6M$)
- $t_1 < \cdots < t_p = T$: fixed-leg payment dates, $\delta_k$: year fraction corresponding to period $(t_{k-1}, t_k)$
- $t = \tilde{t}_0 < \tilde{t}_1 < \cdots < \tilde{t}_q = T$: floating-leg payment dates, $\tilde{\delta}_i$: year fraction of $(\tilde{t}_{i-1}, \tilde{t}_i)$

$$S_{\text{IRS}} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) = \sum_{i=1}^{q} P^D(t_0, \tilde{t}_i) \tilde{\delta}_i F_j(t_0, \tilde{t}_i)$$

where $F_j(t_0, \tilde{t}_i)$ is the forward Libor or Euribor rate defined as the fixed rate to be exchanged at time $\tilde{t}_i$ against the $j$-tenor Libor or Euribor rate established at time $\tilde{t}_{i-1}$ so that the swap has zero value at time $t_0$
Example 4 : credit curve based on CDS

- $S^{\text{CDS}}$: fair spread at time $t_0$ of a credit default swap with maturity $T$
- $t_1 < \cdots < t_p = T$: premium payment dates, $\delta_k$: year fraction corresponding to period $(t_{k-1}, t_k)$
- $R$: expected recovery rate of the reference entity

$$S^{\text{CDS}} \sum_{k=1}^{P} \delta_k P^D(t_0, t_k) Q(t_0, t_k) = -(1 - R) \int_{t_0}^{T} P^D(t_0, u) dQ(t_0, u)$$

where $u \rightarrow Q(t_0, u)$ is the $\mathcal{F}_{t_0}$-conditional (risk-neutral) survival distribution of the reference entity.

We implicitly assume here that recovery, default and interest rates are stochastically independent.
Example 4: credit curve based on CDS (cont)

Using an integration by parts, the survival function \( u \to Q(t_0, u) \) satisfies a linear relation:

\[
S_{\text{CDS}} \sum_{k=1}^{P} \delta_k P^D(t_0, t_k)Q(t_0, t_k) + (1 - R)P^D(t_0, T)Q(t_0, T)
\]

\[
+ (1 - R) \int_{t_0}^{T} f^D(t_0, u)P^D(t_0, u)Q(t_0, u)du = 1 - R
\]

where \( f^D(t_0, u) \) is the instantaneous forward (discount) rate associated with maturity date \( u \).
Geometric nature of the problem

Proposition (admissible curves form a convex set)
Under Assumption I, the set of admissible yield-curve is convex.

This derives immediately from the definition of admissible curves and the linear representation of present values.

Proposition
Under Assumption I, the set of admissible yield-curves is characterized by the convex hull of the extreme points of its closure.

Identifying the set of admissible yield-curves amounts to identify its convex hull.
The proof follows from successive applications of Ascoli-Arzelà theorem and Krein-Milman theorem.

**Ascoli-Arzelà theorem**

Let \((X, d)\) be a compact space. A subset \(F\) of \(C(X)\) is relatively compact if and only if \(F\) is equibounded and equicontinuous.

We have to prove that \(F\) is equibounded and equicontinuous.

**Krein-Milman theorem**

Let \(X\) be a locally convex topological vector space (assumed to be Hausdorff or separable), and let \(K\) be a compact convex subset of \(X\). Then \(K\) is the closed convex hull of its extreme points.
We observe OIS par rates $S_1, \cdots, S_n$ for maturities $T_1 < \cdots < T_n$.

Let $t = t_0 < t_1 < \cdots < t_{p_n} = T_n$ be the annual time grid up to time $T_n$.

The set of indices $(p_i)$ is such that $t_{p_i} = T_i$ for $i = 1, \ldots, n$.

$$S_i \sum_{k=1}^{p_i-1} \delta_k P^D(t_0, t_k) + (S_i \delta_{p_i} + 1) P^D(t_0, T_i) = 1, \quad i = 1, \ldots, n$$

Let $i_0$ be the smallest index such that $T_{i_0} \neq t_{i_0}$ ($i_0 = 11$ in our applications)

Define $H_i := \sum_{k=p_i-1+1}^{p_i-1} \delta_k$, for $i = i_0, \ldots, n$
Proposition (arbitrage-free bounds for discount factors)

\[
P^D(t_0, T_1) = \frac{1}{1 + S_1\delta_1},
\]

\[
P^D(t_0, T_i) = \frac{1}{1 + S_i\delta_i} \left( 1 - \frac{S_i}{S_{i-1}} \left( 1 - P^D(t_0, T_{i-1}) \right) \right), \quad i = 2, \ldots, i_0 - 1
\]

For \( i = i_0, \ldots, n \),

\[
P^D_{\text{min}}(t_0, T_i) \leq P^D(t_0, T_i) \leq P^D_{\text{max}}(t_0, T_i)
\]

where

\[
P^D_{\text{min}}(t_0, T_i) = \frac{1}{1 + S_i\delta_{p_i}} \left( 1 - \frac{S_i}{S_{i-1}} \left( 1 - (1 - S_{i-1}H_i)P^D(t_0, T_{i-1}) \right) \right)
\]

\[
P^D_{\text{max}}(t_0, T_i) = \frac{1}{1 + S_i(H_i + \delta_{p_i})} \left( 1 - \frac{S_i}{S_{i-1}} \left( 1 - P^D(t_0, T_{i-1}) \right) \right)
\]
Proof:

For any \( i = i_0, \ldots, n \), the previous rectangular system of OIS present values can be simplified:

\[
\frac{S_i}{S_{i-1}} \left(1 - P^D(t_0, T_{i-1})\right) + S_i \sum_{k=p_{i-1}+1}^{p_i-1} \delta_k P^D(t_0, t_k) + (1 + S_i \delta_{p_i}) P^D(t_0, T_i) = 1
\]

The bounds derive from the following system of arbitrage-free inequalities:

\[
\begin{cases}
P^D(t_0, T_{i_0}) \leq P^D(t_0, t_k) \leq P^D(t_0, T_{i_0-1}) & \text{for } p_{i_0-1} + 1 \leq k \leq p_{i_0} - 1 \\
\vdots
\end{cases}
\]

\[
P^D(t_0, T_i) \leq P^D(t_0, t_k) \leq P^D(t_0, T_{i-1}) & \text{for } p_{i-1} + 1 \leq k \leq p_i - 1
\]

These bounds cannot be computed since we do not know the discount factors \( P^D(t_0, T_i) \) for \( i = i_0, \ldots, n \).
Iterative computation of model-free bounds

**Step 1**: For \( i = 1, \ldots, i_0 - 1, \)

\[
P^D(t_0, T_i) = \frac{1}{1 + S_i \delta_i} \left( 1 - \frac{S_i}{S_{i-1}} \left( 1 - P^D(t_0, T_{i-1}) \right) \right)
\]

**Step 2**: For \( i = i_0, \ldots, n, \)

\[
P_{\text{min}}(T_i) \leq P^D(t_0, T_i) \leq P_{\text{max}}(T_i)
\]

where

\[
P_{\text{min}}(T_i) = \frac{1}{1 + S_i \delta_{p_i}} \left( 1 - \frac{S_i}{S_{i-1}} \left( 1 - (1 - S_{i-1} H_i) P_{\text{min}}(T_{i-1}) \right) \right)
\]

\[
P_{\text{max}}(T_i) = \frac{1}{1 + S_i (H_i + \delta_{p_i})} \left( 1 - \frac{S_i}{S_{i-1}} \left( 1 - P_{\text{max}}(T_{i-1}) \right) \right)
\]
Arbitrage-free bounds for OIS discount curves

The previous model-free bounds are sharp

Input data: OIS swap rates as of May, 31st 2013
Corresponding model-free bounds on discount rates

Input data: OIS swap rates as of May, 31st 2013
Arbitrage-free bounds for OIS discount curves

Range of arbitrage-free market-consistent OIS discount curves

Input data: OIS swap rates as of May, 31st 2013
Proposition (detecting arbitrage opportunities)

An arbitrage opportunity can be detected in the data set \((S_i)_{i=1,\ldots,n}\) at the first index \(i\) such that

\[
S_i < \left( \frac{1}{S_{i-1}} + \delta_i \frac{P^D(t_0, T_{i-1})}{1 - P^D(t_0, T_{i-1})} \right)^{-1}, \quad i = 2, \ldots, i_0 - 1,
\]

\[
S_i < \left( \frac{1}{S_{i-1}} + (H_i + \delta_{p_i}) \frac{P_{\max}(T_{i-1})}{1 - P_{\max}(T_{i-1})} \right)^{-1}, \quad i = i_0, \ldots, n.
\]

Proof:

For \(i = 2, \ldots, i_0 - 1\), the inequality on \(S_i\) leads to \(P^D(t_0, T_i) > P^D(t_0, T_{i-1})\)

For \(i = i_0, \ldots, n\), the inequality on \(S_i\) leads to \(P^D_{\min}(t_0, T_i) > P^D_{\max}(t_0, T_i)\)
Corollary (increasing OIS par rates are arbitrage-free)

An increasing sequence of OIS par rates $S_1 \leq \cdots \leq S_n$ is arbitrage-free: there always exits an arbitrage-free discount curve which is compatible with this sequence.
We observe **CDS fair spreads** $S_1, \ldots, S_n$ for maturities $T_1 < \cdots < T_n$.

Let $t = t_0 < t_1 < \cdots < t_{pn} = T_n$ be the time grid corresponding to premium payment dates.

The set of indices $(p_i)$ is such that $p_0 = 1$ and $t_{p_i} = T_i$ for $i = 1, \ldots, n$.

For $i = 1, \ldots, n$,

$$S_i \sum_{k=1}^{p_i} \delta_k P^D(t_0, t_k) Q(t_0, t_k) + (1 - R) P^D(t_0, T) Q(t_0, T)$$

$$+ (1 - R) \int_{t_0}^{T_i} f^D(t_0, t) P^D(t_0, t) Q(t_0, t) dt = 1 - R$$
Proposition (arbitrage-free bounds for survival probabilities)

For $i = 1, \ldots, n$,

$$Q_{\text{min}}(t_0, T_i) \leq Q(t_0, T_i) \leq Q_{\text{max}}(t_0, T_i)$$

where

$$Q_{\text{min}}(t_0, T_i) = \frac{1 - R - \sum_{k=1}^{i} ((1 - R)M_k + S_i N_k) Q(t_0, T_{k-1})}{P^D(t_0, T_i)(1 - R + S_i \delta_{p_i})}$$

$$Q_{\text{max}}(t_0, T_i) = \frac{1 - R - \sum_{k=1}^{i-1} ((1 - R)M_k + S_i N_k) Q(t_0, T_k)}{P^D(t_0, T_{i-1})(1 - R) + S_i (N_i + \delta_{p_i}P^D(t_0, T_i))}$$

with $M_i := P^D(t_0, T_{i-1}) - P^D(t_0, T_i)$ and $N_i := \sum_{k=p_i-1}^{p_i-1} \delta_k P^D(t_0, t_k)$. 

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Proof:

For any $i = 1, \ldots, n$, the proof is based on CDS present value representations as linear combinations of survival probabilities and application of the following system of “arbitrage-free inequalities”:

$$
\begin{aligned}
Q(t_0, T_1) &\leq Q(t_0, t) \leq 1 & &\text{for } t_0 \leq t < T_1, \\
& \quad \vdots \\
Q(t_0, T_i) &\leq Q(t_0, t) \leq Q(t_0, T_{i-1}) & &\text{for } T_{i-1} \leq t < T_i
\end{aligned}
$$

These bounds cannot be computed explicitly since we do not know the survival probabilities $Q(t, T_i)$ with certainty for $i = 1, \ldots, n$. 
Iterative computation of model-free bounds

- For $i = 1, \ldots, n$, compute recursively

\[
Q_{\min}(T_i) \leq Q(t_0, T_i) \leq Q_{\max}(T_i)
\]

where

\[
Q_{\min}(T_i) = \frac{1 - R - \sum_{k=1}^{i} ((1 - R)M_k + S_i N_k) Q_{\max}(T_{k-1})}{P^D(t, T_i)(1 - R + S_i \delta_{p_i})}
\]

\[
Q_{\max}(T_i) = \frac{1 - R - \sum_{k=1}^{i-1} ((1 - R)M_k + S_i N_k) Q_{\min}(T_k)}{P^D(t, T_{i-1})(1 - R) + S_i (N_i + \delta_{p_i} P^D(t, T_i))}
\]
Input data: CDS spreads of AIG as of December 17, 2007, $R = 40\%$,

$$P^D(t_0, t) = \exp(-3\%(t - t_0))$$
Bounds sensitivity with respect to the recovery rate assumption

Bounds at standard maturities as a function of recovery rate

Input data: CDS spreads of AIG as of December 17, 2007,
\[ P^D(t_0, t) = \exp(-3\%(t - t_0)) \]
The yield-curve is built from market quotes of a set of standard products:

- $t_0$ : quotation date
- $T = (T_1, \ldots, T_n)$ : set of increasing standard maturities, $T_0 = t_0$
- $S = (S_1, \ldots, S_n)$ : corresponding set of market quotes at $t_0$

We assume that present values can be expressed as linear combination of generic elementary quantities comparable to discount factors:

- $P = P^B$, zero-coupon prices as in Example 1
- $P = P^D$, discount factors as in Example 2
- $P = Q$, risk-neutral survival probabilities as in Example 4

In this presentation, we do not treat the case of elementary quantities comparable to forward rates as in Example 3.
Mean-reverting term-structure models as generators of admissible yield curves

The risk-neutral dynamics of (default-free) interest rates or of default intensities is assumed to follow either

- a OU process driven by a Lévy process

\[ dX_t = a(b(t; p, T, S) - X_t)dt + \sigma dY_{ct}, \]

where \( Y \) is a Lévy process with cumulant function \( \kappa \) and parameter set \( p_L \)

- or an extended CIR process

\[ dX_t = a(b(t; p, T, S) - X_t)dt + \sigma \sqrt{X_t} dW_t, \]

where \( W \) is a standard Browian motion

Depending on the context, \( p = (X_0, a, \sigma, c, p_L) \) will denote the parameter set of the Lévy-OU process and \( p = (X_0, a, \sigma) \) the parameter set of the CIR process
How to construct admissible yield curves?

In both cases, $b$ is represented by a step function:

$$b(t; p, T, S) = b_i(p, T, S) \quad \text{for} \quad T_{i-1} < t \leq T_i, \quad i = 1, \ldots, n$$

The vector $b = (b_1, \ldots, b_n)$ solves the following pseudo-linear system.

**Market-fit linear conditions**

The market-fit condition can be restated as a pseudo-linear system

$$A \cdot P(b) = B$$

where

- $P(b) = (P(t_0, t_k; b))_{k=1,\ldots,m}$ is the $m \times 1$ vector of elementary quantities that appear in the present value formula of instruments used to build the curve (see Examples 1 to 4).
- $A$ is a $n \times m$ matrix, $B$ is a $n \times 1$ matrix
- $A$ and $B$ only depend on current market quotes $S$, on standard maturities $T$ and on products characteristics.
How to construct admissible yield curves?

Proposition (Discount factors in the Lévy-OU approach)

Let $T_{i-1} < t \leq T_i$. In the Lévy-OU model, the current value of the discount factor or of an assimilated quantity with maturity time $t$ is given by

$$P(t_0, t; b) := \mathbb{E} \left[ \exp \left( - \int_{t_0}^{t} X_u du \right) \right] = \exp (-I(t_0, t, b))$$

where

$$I(t_0, t, b) := X_0 \varphi(t - t_0) + \sum_{k=1}^{i-1} b_k (\xi(t - T_{k-1}) - \xi(t - T_k))$$

$$+ b_i \xi(t - T_{i-1}) + c \psi(t - t_0)$$

and functions $\varphi$, $\xi$ and $\psi$ are defined by

$$\varphi(s) := \frac{1}{a} \left( 1 - e^{-as} \right)$$

$$\xi(s) := s - \varphi(s)$$

$$\psi(s) := - \int_{0}^{s} \kappa(-\sigma \varphi(s - \theta)) d\theta$$
How to construct admissible yield curves?

**Proposition (Discount factors in the CIR approach)**

Let $T_{i-1} < t \leq T_i$. In the CIR model, the current value of the discount factor or of an assimilated quantity with maturity time $t$ is given by

$$P(t_0, t; b) := \mathbb{E} \left[ \exp \left( - \int_{t_0}^{t} X_u du \right) \right] = \exp (-I(t_0, t, b))$$

where

$$I(t_0, t, b) := X_0 \varphi(t-t_0) + \sum_{k=1}^{i-1} b_k (\eta(t-T_{k-1}) - \eta(t-T_k)) + b_i \eta(t-T_{i-1})$$

and functions $\varphi$ and $\eta$ are defined by

$$\varphi(s) := \frac{2(1 - e^{-hs})}{h + a + (h - a)e^{-hs}}$$

$$\eta(s) := 2a \left[ \frac{s}{h + a} + \frac{1}{\sigma^2} \log \left( \frac{h + a + (h - a)e^{-hs}}{2h} \right) \right]$$

where $h := \sqrt{a^2 + 2\sigma^2}$.
How to construct admissible yield curves?

Construction of \((b_1, \ldots, b_n)\) by a bootstrap procedure

For any \(i = 1, \ldots, n\), the present value of the instrument with maturity \(T_i\)

- only depends on \(b_1, \ldots, b_i\)
- is a monotonic function with respect to \(b_i\)

The vector \(b = (b_1, \ldots, b_n)\) satisfies a triangular system of non-linear equations that can be solved recursively:

- Find \(b_1\) as the solution of
  \[
  \sum_{j=1}^{p_1} A_{1j} P(t_0, t_j; b_1) = B_1
  \]

- Assume \(b_1, \ldots, b_{k-1}\) are known, find \(b_k\) as the solution of
  \[
  \sum_{j=1}^{p_k} A_{kj} P(t_0, t_j; b_1, \ldots, b_k) = B_k
  \]
How to construct admissible yield curves?

**Proposition (smoothness condition)**

A curve \( t \to P(t_0, t) \) constructed from the previous approach satisfies the smoothness condition: it is of class \( C^1 \) and the corresponding forward curve (or default density function) is continuous.

**Proof:** Let \( b(\cdot) \) be a deterministic function of time, instantaneous forward rates are such that

- Lévy-driven OU

\[
f(t_0, t) = X_0 e^{-a(t-t_0)} + a \int_{t_0}^{t} e^{-a(t-u)} b(u) du - c\kappa(-\sigma \varphi(t - t_0))
\]

where \( \varphi \) is defined by (1)

- extended CIR

\[
f^{CIR}(t_0, t) = X_0 \varphi'(t - t_0) + a \int_{t_0}^{t} \varphi'(t - u) b(u) du
\]

where \( \varphi' \) is the derivative of \( \varphi \) given by (2)
How to construct admissible yield curves?

Assume that a curve has been constructed from a **Lévy-OU term-structure model** with positive parameters \((X_0, a, \sigma, c, p_L)\):

\[
f(t_0, t) = X_0 e^{-a(t-t_0)} + a \sum_{k=1}^{i-1} b_k (\varphi(t - T_{k-1}) - \varphi(t - T_k))
\]

\[
+ ab_i \varphi(t - T_{i-1}) - c\kappa(-\sigma \varphi(t - t_0))
\]

for any \(T_{i-1} \leq t \leq T_i, \ i = 1, \ldots, n\).

**Proposition (arbitrage-free condition in the Lévy-OU approach)**

Assume that the derivative of the Lévy cumulant \(\kappa'\) exists and is strictly monotonic on \((-\infty, 0)\). The curve is arbitrage-free on the time interval \((t_0, T_n)\) if and only if, for any \(i = 1, \ldots, n\), \(f(t_0, T_i) > 0\) and one of the following condition holds:

- \(\frac{\partial f}{\partial t}(t_0, T_{i-1}) \frac{\partial f}{\partial t}(t_0, T_i) \geq 0\)
- \(\frac{\partial f}{\partial t}(t_0, T_{i-1}) \frac{\partial f}{\partial t}(t_0, T_i) < 0\) and \(f(t_0, t_i) > 0\) where \(t_i\) is such that \(\frac{\partial f}{\partial t}(t_0, t_i) = 0\).
Assume that a curve has been constructed from an extended CIR term-structure model with positive parameters \((X_0, a, \sigma)\):

\[
f^{CIR}(t_0, t) = X_0 \varphi'(t-t_0) + a \sum_{k=1}^{i-1} b_k (\varphi(t - T_{k-1}) - \varphi(t - T_k)) + ab_i \varphi(t - T_{i-1})
\]

for any \(T_{i-1} \leq t \leq T_i, \ i = 1, \ldots, n\).

**Proposition (arbitrage-free condition in the Lévy-OU approach)**

The constructed curve is arbitrage-free if, for any \(i = 1, \ldots, n\), the implied \(b_i\) is positive.
Set of admissible OIS discount and forward curves: Lévy-OU short rates

Parameters: \( a = 0.01, \sigma = 1, X_0 = 0.063\% \) (fair rate of IRS vs OIS 1M). The Lévy driver is a Gamma subordinator with parameter \( \lambda = 1/50\text{bps} \) (mean jump size of 50 bps). \( c = \{1, 10, 20, \ldots, 100\} \)

Input data: OIS swap rates as of May, 31st 2013
How to construct admissible yield curves?

Arbitrage-free bounds used to generate a wider range of admissible curves

Parameters: CIR short rates with $a = 5$, $\sigma = 1$, $X_0 = 0.063\%$ (fair rate of IRS vs OIS 1M).

Input data: OIS swap rates as of May, 31st 2013
How to construct admissible yield curves?

Set of admissible survival curves: CIR intensities

Parameters: $a = \sigma = 1$, $100X_0 = \{0.01, 0.25, 0.49, 0.73, 0.97, 1.21, 1.45, 1.69, 1.94, 2.18, 2.42\}$

Input data: CDS spreads of AIG as of December 17, 2007, $R = 40\%$, $P^D(t_0, t) = \exp(-3\%(t - t_0))$
Set of admissible survival curves: Lévy-OU intensities

Parameters: \( a \) is Uniform on \([0.5, 10]\), \( c \) is Uniform on \([1, 50]\) (\( c \): mean number of jumps per year), \( \sigma = 1 \). The Lévy driver is a Gamma subordinator with parameter \( \lambda = 1/2 \) bps (mean jump size of 2 bps), \( X_0 \) is bootstrapped with \( b_1 \) in such a way that \( X_0 = b_1 \)

Input data: CDS spreads of AIG as of December 17, 2007
The proposed framework could be extended or used in several directions:

1. **Yield-curve diversity impact on present values (PV) and hedging strategies?**
   \[
   \max_{i,j} \| PV(C_i) - PV(C_j) \|_p
   \]
   where the max is taken over all couples of admissible curves \((C_i, C_j)\).

2. **Sensitivity analysis in the presence of uncertain parameters?**
   \[
   dX_t = \tilde{a}(b(t; \tilde{a}, \tilde{\sigma}, T, S) - X_t)dt + \tilde{\sigma}\sqrt{X_t}dW_t,
   \]
   where \(\text{Range}(\tilde{a}, \tilde{\sigma}) \subset \{(a, \sigma) \mid b(t; a, \sigma, T, S) \geq 0 \ \forall t\}\).

3. **Extension to a multicurve environment?**

4. **Impact on the assessment of counterparty credit risk (CVA, EE, EPE, ...)?**
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Cumulant function of some Lévy processes

<table>
<thead>
<tr>
<th>Process</th>
<th>Cumulant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brownian motion</td>
<td>$\kappa(\theta) = \frac{\theta^2}{2}$</td>
</tr>
<tr>
<td>Gamma process</td>
<td>$\kappa(\theta) = -\log \left(1 - \frac{\theta}{\lambda}\right)$</td>
</tr>
<tr>
<td>Inverse Gaussian process</td>
<td>$\kappa(\theta) = \lambda - \sqrt{\lambda^2 - 2\theta}$</td>
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</tbody>
</table>