Model risk embedded in yield curve construction methods

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Introduction

- What is understood as a term-structure in this presentation ?
- What is it used for ?

Introduction

- Term-structures are constructed from market quotes of fixed income, fx or default-risky securities
- Information provided by the market is reliable only for a small set of liquid products with standard characteristics/maturities
- We have to rely on interpolation/calibration schemes to construct the curve for missing maturities
- A variety of curve construction methods exists : no consensus towards a particular best practice in all circumstances

Andersen (2007), curves based on tension splines

Figure 1: Yield Curve

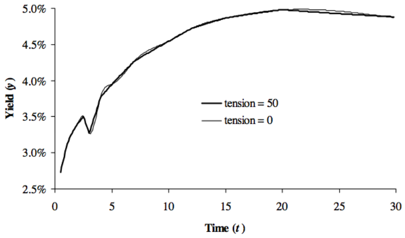
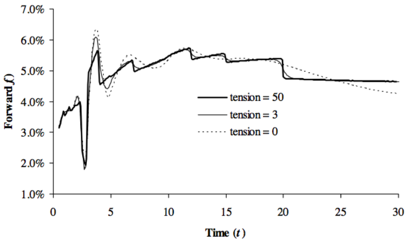
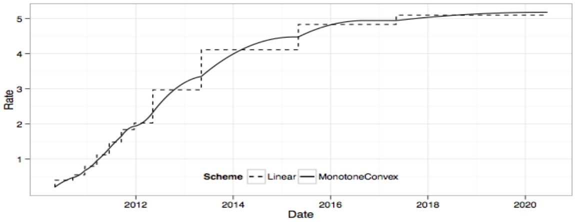


Figure 2: Forward Curve



Le Floc'h (2012), examples of one-day forward curves



What can be defined as a good yield curve construction method ? (Hagan and West (2006))

- Ability to fit market quotes
- Arbitrage freeness
- Smoothness
- Locality of the interpolation method
- Stability of forward rate
- Consistency of hedging strategies : Locality of deltas ? Sum of sequential deltas close enough to the corresponding parallel delta ? (Le Floc'h (2012))

Interestingly, there is a pretty large recent literature on the subject of yield-curve construction methods

- Single-curve environment : Hagan and West (2006), Andersen (2007), Jerassy-Etzion (2010), Le Floc'h (2012)
- Multi-curve environment : Ametrano and Bianchetti (2009), Iwashita (2013), Kenyon and Stamm (2012), Fries (2013), Chibane et al (2009)

And a flourishing literature on model risk

- Branger and Schlag (2004), Cont (2006), Davis and Hobson (2004), Derman (1996), Eberlein and Jacod (1997), El Karoui et al (1998), Green and Figlewski (1999), Hénaff (2010), Morini (2010), etc...

Arbitrage-free curve

A curve is said to be arbitrage-free if

- IR curves : the forward rates are non-negative or equivalently, the (pseudo) discount factors are nonincreasing with respect to time-to-maturities
- Credit : the curve is associated with a well-defined default distribution function

Smoothness condition

A curve is said to be smooth if

- IR curves : the instantaneous forward rates exist for all maturities and are continuous.
- Credit : the default density function exists and is continuous.

Admissible curve

A yield curve is said to be admissible if it satisfies the following constraints :

- The input data set is perfectly reproduced by the curve
- The curve is arbitrage-free
- The curve satisfies the smoothness condition

We then address the following questions :

- Is it possible to estimate the size of admissible curves ? and how ?
- How does the range/diversity of admissible curves affect the present value of products with non-standard characteristics ?

We develop a framework in which it is possible to measure the diversity of yield curves with some specific features.

Instruments used for curve construction

Assumption I : pseudo-linear representation of present values

Products used in the curve construction have presents values that can be expressed as linear combination of some elementary quantities such as zero-coupon prices, discount factors, Ibor forward rates or survival probabilities.

Example 1 : Corporate or sovereign debt yield curve

- S : market price (in percentage of nominal) at time t_0 of a bond with maturity T
- c : fixed coupon rate
- $t_1 < \ldots < t_p = T$: coupon payment dates, δ_k : year fraction corresponding to period (t_{k-1}, t_k)

$$c \sum_{k=1}^{p} \delta_k P^B(t_0, t_k) + P^B(t_0, T) = S$$

where $P^B(t_0, t_k)$ represents the price of a (fictitious default-free issuer-dependent) ZC bond with maturity t_k

Example 2 : Discounting curve based on OIS

- S^{OIS} : par rate at time t_0 of an overnight indexed swap with maturity T
- $t_1 < \cdots < t_p = T$: fixed-leg payment dates
- δ_k : year fraction corresponding to period (t_{k-1}, t_k)

$$S^{\text{OIS}} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) = 1 - P^D(t_0, T)$$

where $P^D(t_0, t_k)$ is the discount factor associated with maturity date t_k

Example 3 : Forward curve based on fixed-vs-Ibor-floating IRS

- S^{IRS} : par rate at time t_0 of an interest rate swap with maturity T and tenor j (typically, $j = 3\text{M}$ or $j = 6\text{M}$)
- $t_1 < \cdots < t_p = T$: fixed-leg payment dates, δ_k : year fraction corresponding to period (t_{k-1}, t_k)
- $t = \tilde{t}_0 < \tilde{t}_1 < \cdots < \tilde{t}_q = T$: floating-leg payment dates, $\tilde{\delta}_i$: year fraction of $(\tilde{t}_{i-1}, \tilde{t}_i)$

$$S^{\text{IRS}} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) = \sum_{i=1}^{q} P^D(t_0, \tilde{t}_i) \tilde{\delta}_i F_j(t_0, \tilde{t}_i)$$

where $F_j(t_0, \tilde{t}_i)$ is the forward Libor or Euribor rate defined as the fixed rate to be exchanged at time $\tilde{t}_i$ against the j-tenor Libor or Euribor rate established at time $\tilde{t}_{i-1}$ so that the swap has zero value at time t_0

Example 4 : credit curve based on CDS

- S^{CDS} : fair spread at time t_0 of a credit default swap with maturity T
- $t_1 < \cdots < t_p = T$: premium payment dates, δ_k : year fraction corresponding to period (t_{k-1}, t_k)
- R : expected recovery rate of the reference entity

$$S^{\text{CDS}} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) Q(t_0, t_k) = -(1 - R) \int_{t_0}^{T} P^D(t_0, u) dQ(t_0, u)$$

where $u \to Q(t_0, u)$ is the $\mathcal{F}_{t_0}$-conditional (risk-neutral) survival distribution of the reference entity.

We implicitly assume here that recovery, default and interest rates are stochastically independent.

Example 4 : credit curve based on CDS (cont)

Using an integration by parts, the survival function $u \to Q(t_0, u)$ satisfies a linear relation :

$$S^{\text{CDS}} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) Q(t_0, t_k) + (1 - R) P^D(t_0, T) Q(t_0, T)$$
$$+ (1 - R) \int_{t_0}^{T} f^D(t_0, u) P^D(t_0, u) Q(t_0, u) du = 1 - R$$

where $f^D(t_0, u)$ is the instantaneous forward (discount) rate associated with maturity date u.

Geometric nature of the problem

Proposition (admissible curves form a convex set)

Under Assumption I, the set of admissible yield-curve is convex.

This derives immediately from the definition of admissible curves and the linear representation of present values.

Proposition

Under Assumption I, the set of admissible yield-curves is characterized by the convex hull of the extreme points of its closure.

Identifying the set of admissible yield-curves amounts to identify its convex hull

Geometric nature of the problem

The proof follows from successive applications of Ascoli-Arzelà theorem and Krein-Milman theorem.

Ascoli-Arzelà theorem

Let (X, d) be a compact space. A subset F of $\mathcal{C}(X)$ is relatively compact if and only if F is equibounded and equicontinuous.

We have to prove that F is equibounded and equicontinuous.

Krein-Milman theorem

Let X be a locally convex topological vector space (assumed to be Hausdorff or separable), and let K be a compact convex subset of X. Then K is the closed convex hull of its extreme points.

Arbitrage-free bounds for OIS discount curves

- We observe OIS par rates $S_1, \cdots, S_n$ for maturities $T_1 < \cdots < T_n$.
- Let $t = t_0 < t_1 < \cdots < t_{p_n} = T_n$ be the annual time grid up to time T_n.
- The set of indices (p_i) is such that $t_{p_i} = T_i$ for $i = 1, ..., n$.

$$S_i \sum_{k=1}^{p_i - 1} \delta_k P^D(t_0, t_k) + (S_i \delta_{p_i} + 1) P^D(t_0, T_i) = 1, \quad i = 1, ..., n$$

- Let i_0 be the smallest index such that $T_{i_0} \neq t_{i_0}$ ($i_0 = 11$ in our applications)
- Define $H_i := \displaystyle\sum_{k=p_{i-1}+1}^{p_i - 1} \delta_k$, for $i = i_0, \ldots, n$

Proposition (arbitrage-free bounds for discount factors)

$$P^D(t_0, T_1) = \frac{1}{1 + S_1\delta_1},$$

$$P^D(t_0, T_i) = \frac{1}{1 + S_i\delta_i}\left(1 - \frac{S_i}{S_{i-1}}\left(1 - P^D(t_0, T_{i-1})\right)\right), \quad i = 2, \ldots, i_0 - 1$$

For $i = i_0, \ldots, n,$

$$P^D_{\min}(t_0, T_i) \leqslant P^D(t_0, T_i) \leqslant P^D_{\max}(t_0, T_i)$$

where

$$P^D_{\min}(t_0, T_i) = \frac{1}{1 + S_i\delta_{p_i}}\left(1 - \frac{S_i}{S_{i-1}}\left(1 - (1 - S_{i-1}H_i)P^D(t_0, T_{i-1})\right)\right)$$

$$P^D_{\max}(t_0, T_i) = \frac{1}{1 + S_i(H_i + \delta_{p_i})}\left(1 - \frac{S_i}{S_{i-1}}\left(1 - P^D(t_0, T_{i-1})\right)\right)$$

Proof :

For any $i = i_0, \ldots, n$, the previous rectangular system of OIS present values can be simplified :

$$\frac{S_i}{S_{i-1}}\Big(1 - P^D(t_0, T_{i-1})\Big) + S_i \sum_{k=p_{i-1}+1}^{p_i-1} \delta_k P^D(t_0, t_k) + (1 + S_i\delta_{p_i})\, P^D(t_0, T_i) = 1$$

The bounds derive from the following system of arbitrage-free inequalities :

$$\begin{cases} P^D(t_0, T_{i_0}) \leqslant P^D(t_0, t_k) \leqslant P^D(t_0, T_{i_0-1}) & \text{for } p_{i_0-1} + 1 \leqslant k \leqslant p_{i_0} - \\ \vdots & \\ P^D(t_0, T_i) \leqslant P^D(t_0, t_k) \leqslant P^D(t_0, T_{i-1}) & \text{for } p_{i-1} + 1 \leqslant k \leqslant p_i - 1 \end{cases}$$

These bounds cannot be computed since we do not know the discount factors $P^D(t_0, T_i)$ for $i = i_0, \ldots, n$

Iterative computation of model-free bounds

- **Step 1** : For $i = 1, \dots, i_0 - 1$,

$$P^D(t_0, T_i) = \frac{1}{1 + S_i \delta_i} \left(1 - \frac{S_i}{S_{i-1}} \left(1 - P^D(t_0, T_{i-1}) \right) \right)$$

- **Step 2** : For $i = i_0, \dots, n$,

$$P_{\min}(T_i) \leqslant P^D(t_0, T_i) \leqslant P_{\max}(T_i)$$

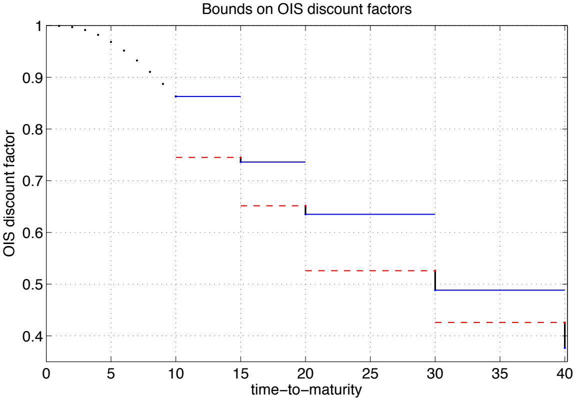
where

$$P_{\min}(T_i) = \frac{1}{1 + S_i \delta_{p_i}} \left(1 - \frac{S_i}{S_{i-1}} (1 - (1 - S_{i-1} H_i) P_{\min}(T_{i-1})) \right)$$

$$P_{\max}(T_i) = \frac{1}{1 + S_i (H_i + \delta_{p_i})} \left(1 - \frac{S_i}{S_{i-1}} (1 - P_{\max}(T_{i-1})) \right)$$

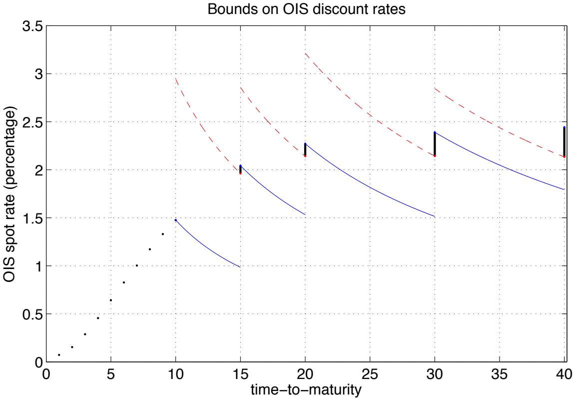
Arbitrage-free bounds for OIS discount curves

The previous model-free bounds are sharp



Input data : OIS swap rates as of May, 31st 2013

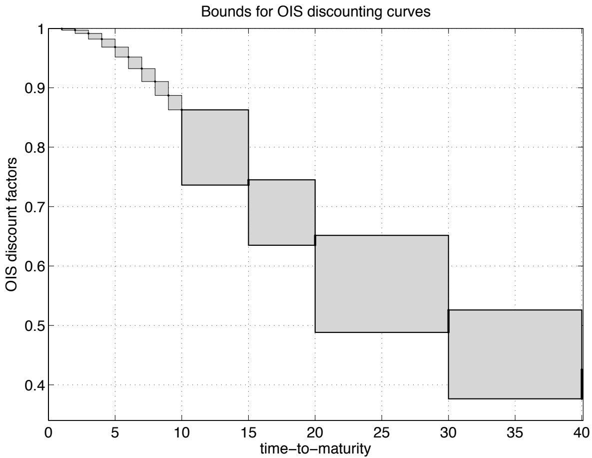
Corresponding model-free bounds on discount rates



Input data : OIS swap rates as of May, 31st 2013

Arbitrage-free bounds for OIS discount curves

Range of arbitrage-free market-consistent OIS discount curves



Input data : OIS swap rates as of May, 31st 2013

Arbitrage-free bounds for OIS discount curves

Proposition (detecting arbitrage opportunities)

An arbitrage opportunity can be detected in the data set $(S_i)_{i=1,\dots,n}$ at the first index i such that

$$S_i < \left(\frac{1}{S_{i-1}} + \delta_i \frac{P^D(t_0, T_{i-1})}{1 - P^D(t_0, T_{i-1})}\right)^{-1}, \; i = 2, \dots, i_0 - 1,$$

$$S_i < \left(\frac{1}{S_{i-1}} + (H_i + \delta_{p_i}) \frac{P_{\max}(T_{i-1})}{1 - P_{\max}(T_{i-1})}\right)^{-1}, \; i = i_0, \dots, n.$$

Proof :

For $i = 2, \dots, i_0 - 1$, the inequality on S_i leads to $P^D(t_0, T_i) > P^D(t_0, T_{i-1})$

For $i = i_0, \dots, n$, the inequality on S_i leads to $P^D_{\min}(t_0, T_i) > P^D_{\max}(t_0, T_i)$

Arbitrage-free bounds for OIS discount curves

Corollary (increasing OIS par rates are arbitrage-free)

An increasing sequence of OIS par rates $S_1 \leq \cdots \leq S_n$ is arbitrage-free : there always exits an arbitrage-free discount curve which is compatible with this sequence.

Arbitrage-free bounds for survival curves

- We observe CDS fair spreads $S_1, \dots, S_n$ for maturities $T_1 < \cdots < T_n$.
- Let $t = t_0 < t_1 < \cdots < t_{p_n} = T_n$ be the time grid corresponding to premium payment dates.
- The set of indices (p_i) is such that $p_0 = 1$ and $t_{p_i} = T_i$ for $i = 1, ..., n$.
- For $i = 1, \dots, n$,

$$S_i \sum_{k=1}^{p_i} \delta_k P^D(t_0, t_k) Q(t_0, t_k) + (1-R) P^D(t_0, T) Q(t_0, T)$$
$$+ (1-R) \int_{t_0}^{T_i} f^D(t_0, t) P^D(t_0, t) Q(t_0, t) dt = 1 - R$$

Arbitrage-free bounds for survival curves

Proposition (arbitrage-free bounds for survival probabilities)

For $i = 1, \ldots, n$,

$$Q_{\min}(t_0, T_i) \leqslant Q(t_0, T_i) \leqslant Q_{\max}(t_0, T_i)$$

where

$$Q_{\min}(t_0, T_i) = \frac{1 - R - \sum_{k=1}^{i} \left((1-R)M_k + S_i N_k\right) Q(t_0, T_{k-1})}{P^D(t_0, T_i)(1 - R + S_i \delta_{p_i})},$$

$$Q_{\max}(t_0, T_i) = \frac{1 - R - \sum_{k=1}^{i-1} \left((1-R)M_k + S_i N_k\right) Q(t_0, T_k)}{P^D(t_0, T_{i-1})(1 - R) + S_i \left(N_i + \delta_{p_i} P^D(t_0, T_i)\right)},$$

with $M_i := P^D(t_0, T_{i-1}) - P^D(t_0, T_i)$ and $N_i := \sum_{k=p_{i-1}}^{p_i - 1} \delta_k P^D(t_0, t_k)$.

Proof :

For any $i = 1, \ldots, n$, the proof is based on CDS present value representations as linear combinations of survival probabilities and application of the following system of "arbitrage-free inequalities" :

$$\begin{cases} Q(t_0, T_1) \leqslant Q(t_0, t) \leqslant 1 & \text{for } t_0 \leqslant t < T_1, \\ \qquad\qquad \vdots & \\ Q(t_0, T_i) \leqslant Q(t_0, t) \leqslant Q(t_0, T_{i-1}) & \text{for } T_{i-1} \leqslant t < T_i \end{cases}$$

These bounds cannot be computed explicitly since we do not know the survival probabilities $Q(t, T_i)$ with certainty for $i = 1, \ldots, n$

Iterative computation of model-free bounds

- For $i = 1, \dots, n$, compute recursively

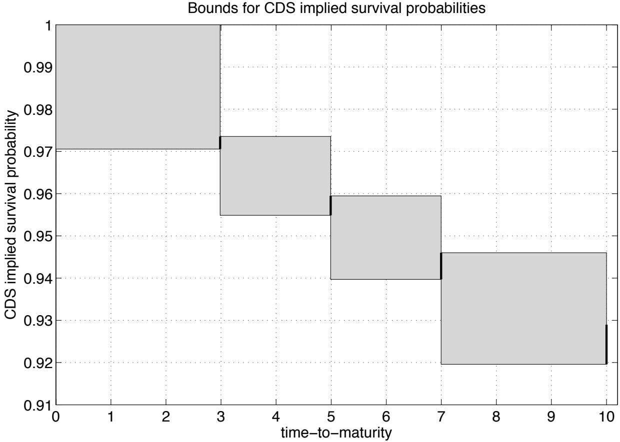
$$Q_{\min}(T_i) \leqslant Q(t_0, T_i) \leqslant Q_{\max}(T_i)$$

where

$$Q_{\min}(T_i) = \frac{1 - R - \sum_{k=1}^{i} \left((1-R)M_k + S_i N_k\right) Q_{\max}(T_{k-1})}{P^D(t, T_i)(1 - R + S_i \delta_{p_i})}$$

$$Q_{\max}(T_i) = \frac{1 - R - \sum_{k=1}^{i-1} \left((1-R)M_k + S_i N_k\right) Q_{\min}(T_k)}{P^D(t, T_{i-1})(1-R) + S_i\left(N_i + \delta_{p_i} P^D(t, T_i)\right)}$$

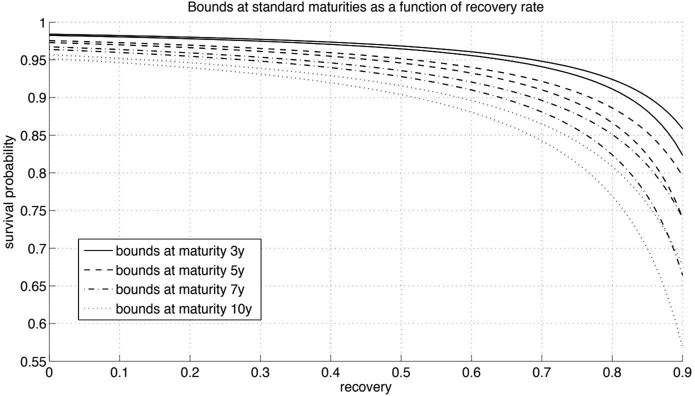
Range of arbitrage-free market-consistent survival curves



Input data : CDS spreads of AIG as of December 17, 2007, $R = 40\%$,

$$P^D(t_0, t) = \exp(-3\%(t - t_0))$$

Bounds sensitivity with respect to the recovery rate assumption



Input data : CDS spreads of AIG as of December 17, 2007,

$$P^D(t_0, t) = \exp(-3\%(t - t_0))$$

How to construct admissible yield curves ?

The yield-curve is built from market quotes of a set of standard products

- t_0 : quotation date
- $\mathbf{T} = (T_1, \ldots, T_n)$: set of increasing standard maturities, $T_0 = t_0$
- $\mathbf{S} = (S_1, \ldots, S_n)$: corresponding set of market quotes at t_0

We assume that present values can be expressed as linear combination of generic elementary quantities comparable to discount factors :

- $P = P^B$, zero-coupon prices as in ▸ Example 1
- $P = P^D$, discount factors as in ▸ Example 2
- $P = Q$, risk-neutral survival probabilities as in ▸ Example 4

In this presentation, we do not treat the case of elementary quantities comparable to forward rates as in ▸ Example 3.

Mean-reverting term-structure models as generators of admissible yield curves

The risk-neutral dynamics of (default-free) interest rates or of default intensities is assumed to follow either

- a OU process driven by a Lévy process

$$dX_t = a(b(t; \mathbf{p}, \mathbf{T}, \mathbf{S}) - X_t)dt + \sigma dY_{ct},$$

  where Y is a Lévy process with cumulant function κ and parameter set $\mathbf{p}_L$

- or an extended CIR process

$$dX_t = a(b(t; \mathbf{p}, \mathbf{T}, \mathbf{S}) - X_t)dt + \sigma\sqrt{X_t}dW_t,$$

  where W is a standard Browian motion

Depending on the context, $\mathbf{p} = (X_0, a, \sigma, c, \mathbf{p}_L)$ will denote the parameter set of the Lévy-OU process and $\mathbf{p} = (X_0, a, \sigma)$ the parameter set of the CIR process

In both cases, b is represented by a step function :

$$b(t;\mathbf{p},\mathbf{T},\mathbf{S}) = b_i(\mathbf{p},\mathbf{T},\mathbf{S}) \text{ for } T_{i-1} < t \leqslant T_i, \quad i = 1,\dots,n$$

The vector $\mathbf{b} = (b_1,\dots,b_n)$ solves the following pseudo-linear system.

Market-fit linear conditions

The market-fit condition can be restated as a pseudo-linear system

$$\mathbf{A} \cdot \mathbf{P}(\mathbf{b}) = \mathbf{B}$$

where

- $\mathbf{P}(\mathbf{b}) = (P(t_0, t_k; \mathbf{b}))_{k=1,\dots,m}$ is the $m \times 1$ vector of elementary quantities that appear in the present value formula of instruments used to build the curve (see Examples 1 to 4).
- $\mathbf{A}$ is a $n \times m$ matrix, $\mathbf{B}$ is a $n \times 1$ matrix
- $\mathbf{A}$ and $\mathbf{B}$ only depend on current market quotes $\mathbf{S}$, on standard maturities $\mathbf{T}$ and on products characteristics.

How to construct admissible yield curves ?

Proposition (Discount factors in the Lévy-OU approach)

Let $T_{i-1} < t \leqslant T_i$. In the Lévy-OU model, the current value of the discount factor or of an assimilated quantity with maturity time t is given by

$$P(t_0, t; \mathbf{b}) := \mathbb{E}\left[\exp\left(-\int_{t_0}^{t} X_u du\right)\right] = \exp\left(-I(t_0, t, \mathbf{b})\right)$$

where

$$\begin{aligned} I(t_0, t, \mathbf{b}) := X_0\varphi(t - t_0) + \sum_{k=1}^{i-1} b_k \left(\xi(t - T_{k-1}) - \xi(t - T_k)\right) \\ + b_i\xi(t - T_{i-1}) + c\psi(t - t_0) \end{aligned}$$

and functions φ, ξ and ψ are defined by

$$\begin{aligned} \varphi(s) &:= \frac{1}{a}\left(1 - e^{-as}\right) \\ \xi(s) &:= s - \varphi(s) \\ \psi(s) &:= -\int_0^s \kappa\left(-\sigma\varphi(s - \theta)\right) d\theta \end{aligned} \tag{1}$$

Proposition (Discount factors in the CIR approach)

Let $T_{i-1} < t \leqslant T_i$. In the CIR model, the current value of the discount factor or of an assimilated quantity with maturity time t is given by

$$P(t_0, t; \mathbf{b}) := \mathbb{E}\left[\exp\left(-\int_{t_0}^{t} X_u du\right)\right] = \exp\left(-I(t_0, t, \mathbf{b})\right)$$

where

$$I(t_0, t, \mathbf{b}) := X_0 \varphi(t - t_0) + \sum_{k=1}^{i-1} b_k \left(\eta(t - T_{k-1}) - \eta(t - T_k)\right) + b_i \eta(t - T_{i-1})$$

and functions φ and η are defined by

$$\begin{aligned} \varphi(s) &:= \frac{2(1 - e^{-hs})}{h + a + (h - a)e^{-hs}} \\ \eta(s) &:= 2a\left[\frac{s}{h + a} + \frac{1}{\sigma^2} \log \frac{h + a + (h - a)e^{-hs}}{2h}\right] \end{aligned} \tag{2}$$

where $h := \sqrt{a^2 + 2\sigma^2}$

Construction of $(b_1, \ldots, b_n)$ by a bootstrap procedure

For any $i = 1, \ldots, n$, the present value of the instrument with maturity T_i

- only depends on $b_1, \ldots, b_i$
- is a monotonic function with respect to b_i

The vector $\mathbf{b} = (b_1, \ldots, b_n)$ satisfies a triangular system of non-linear equations that can be solved recursively :

- Find b_1 as the solution of

$$\sum_{j=1}^{p_1} \mathbf{A}_{1j} P(t_0, t_j; b_1) = \mathbf{B}_1$$

- Assume $b_1, \ldots, b_{k-1}$ are known, find b_k as the solution of

$$\sum_{j=1}^{p_k} \mathbf{A}_{kj} P(t_0, t_j; b_1, \ldots, b_k) = \mathbf{B}_k$$

How to construct admissible yield curves ?

Proposition (smoothness condition)

A curve $t \to P(t_0, t)$ constructed from the previous approach satisfies the smoothness condition : it is of class $\mathcal{C}^1$ and the corresponding forward curve (or default density function) is continuous.

Proof : Let $b(\cdot)$ be a deterministic function of time, instantaneous forward rates are such that

- Lévy-driven OU

$$f(t_0, t) = X_0 e^{-a(t-t_0)} + a \int_{t_0}^{t} e^{-a(t-u)} b(u) du - c\kappa(-\sigma\varphi(t - t_0))$$

  where φ is defined by (1)

- extended CIR

$$f^{CIR}(t_0, t) = X_0 \varphi'(t - t_0) + a \int_{t_0}^{t} \varphi'(t - u) b(u) du$$

  where φ' is the derivative of φ given by (2)

How to construct admissible yield curves ?

Assume that a curve has been constructed from a Lévy-OU term-structure model with positive parameters $(X_0, a, \sigma, c, \mathbf{p}_L)$:

$$f(t_0, t) = X_0 e^{-a(t-t_0)} + a \sum_{k=1}^{i-1} b_k \left(\varphi(t - T_{k-1}) - \varphi(t - T_k)\right) + ab_i\varphi(t - T_{i-1}) - c\kappa(-\sigma\varphi(t - t_0))$$

for any $T_{i-1} \leq t \leq T_i$, $i = 1, \ldots, n$.

Proposition (arbitrage-free condition in the Lévy-OU approach)

Assume that the derivative of the Lévy cumulant κ' exists and is strictly monotonic on $(-\infty, 0)$. The curve is arbitrage-free on the time interval (t_0, T_n) if and only if, for any $i = 1, \ldots, n$, $f(t_0, T_i) > 0$ and one of the following condition holds :

- $\frac{\partial f}{\partial t}(t_0, T_{i-1})\frac{\partial f}{\partial t}(t_0, T_i) \geq 0$
- $\frac{\partial f}{\partial t}(t_0, T_{i-1})\frac{\partial f}{\partial t}(t_0, T_i) < 0$ and $f(t_0, t_i) > 0$ where t_i is such that $\frac{\partial f}{\partial t}(t_0, t_i) = 0$,

How to construct admissible yield curves ?

Assume that a curve has been constructed from a extended CIR term-structure model with positive parameters (X_0, a, σ) :

$$f^{CIR}(t_0, t) = X_0\varphi'(t - t_0) + a\sum_{k=1}^{i-1} b_k \left(\varphi(t - T_{k-1}) - \varphi(t - T_k)\right) + ab_i\varphi(t - T_{i-1})$$

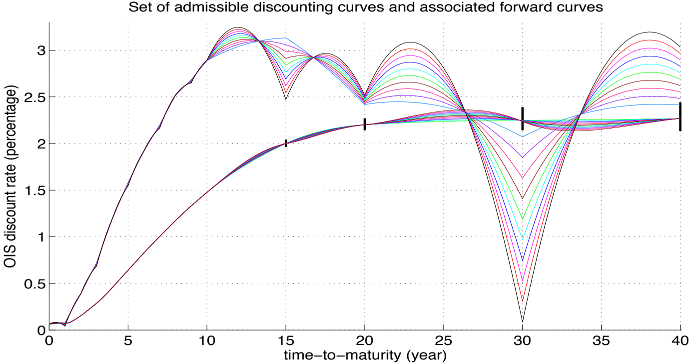
for any $T_{i-1} \leq t \leq T_i$, $i = 1, \ldots, n$.

Proposition (arbitrage-free condition in the Lévy-OU approach)

The constructed curve is arbitrage-free if, for any $i = 1, \cdots, n$, the implied b_i is positive

Set of admissible OIS discount and forward curves : Lévy-OU short rates

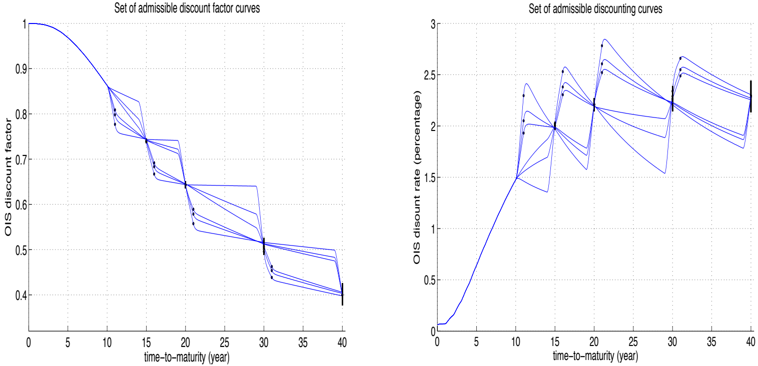
Parameters : $a = 0.01$, $\sigma = 1$, $X_0 = 0.063\%$ (fair rate of IRS vs OIS 1M). The Lévy driver is a Gamma subordinator with parameter $\lambda = 1/50\text{bps}$ (mean jump size of 50 bps). $c = \{1, 10, 20, \ldots, 100\}$



Input data : OIS swap rates as of May, 31st 2013

Arbitrage-free bounds used to generate a wider range of admissible curves

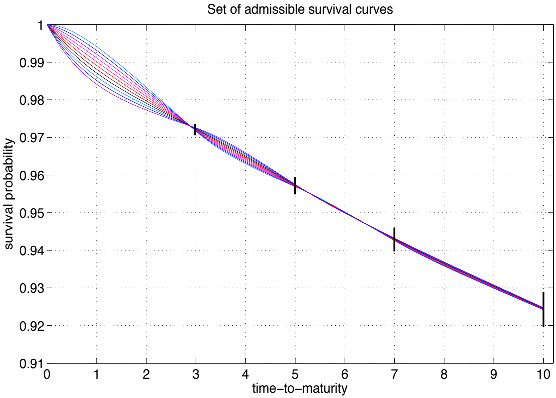
Parameters : CIR short rates with $a = 5$, $\sigma = 1$, $X_0 = 0.063\%$ (fair rate of IRS vs OIS 1M).



Input data : OIS swap rates as of May, 31st 2013

Set of admissible survival curves : CIR intensities

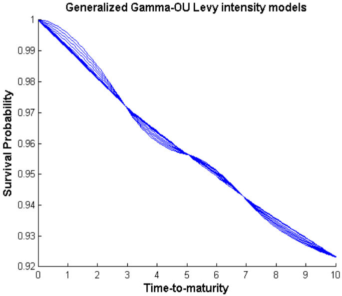
Parameters : $a = \sigma = 1$, $100X_0 = \{0.01, 0.25, 0.49, 0.73, 0.97, 1.21, 1.45, 1.69, 1.94, 2.18, 2.42\}$

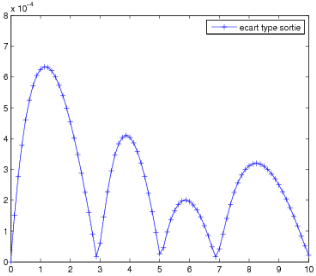


Input data : CDS spreads of AIG as of December 17, 2007, $R = 40\%$,
$$P^D(t_0, t) = \exp(-3\%(t - t_0))$$

Set of admissible survival curves : Lévy-OU intensities

Parameters : a is Uniform on $[0.5, 10]$, c is Uniform on $[1, 50]$ (c : mean number of jumps per year), $\sigma = 1$. The Lévy driver is a Gamma subordinator with parameter $\lambda = 1/2$bps (mean jump size of 2 bps), X_0 is bootstrapped with b_1 in such a way that $X_0 = b_1$





Input data : CDS spreads of AIG as of December 17, 2007

Perspectives

The proposed framework could be extended or used in several directions :

- Yield-curve diversity impact on present values (PV) and hedging stategies ?

$$\max_{i,j} \|PV(C_i) - PV(C_j)\|_p$$

  where the max is taken over all couples of admissible curves (C_i, C_j)

- Sensitivity analysis in the presence of uncertain parameters ?

$$dX_t = \tilde{a}(b(t; \tilde{a}, \tilde{\sigma}, \mathbf{T}, \mathbf{S}) - X_t)dt + \tilde{\sigma}\sqrt{X_t}dW_t,$$

  where $\mathrm{Range}(\tilde{a}, \tilde{\sigma}) \subset \{(a, \sigma) \mid b(t; a, \sigma, \mathbf{T}, \mathbf{S}) \geq 0 \ \forall t\}$

- Extension to a multicurve environment ?

- Impact on the assessment of counterparty credit risk (CVA, EE, EPE, ...) ?

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Cumulant function of some Lévy processes

	Cumulant
Brownian motion	$\kappa(\theta) = \frac{\theta^2}{2}$
Gamma process	$\kappa(\theta) = -\log\left(1 - \frac{\theta}{\lambda}\right)$
Inverse Gaussian process	$\kappa(\theta) = \lambda - \sqrt{\lambda^2 - 2\theta}$