Hedging default risks of CDOs in Markovian contagion models

Areski COUSIN

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Introduction

- **Purpose of the paper**
  - Describe a hedging strategy of CDO tranches
  - Based upon dynamic trading of corresponding CDS Index and the risk-free asset

- **Contagion models**
  - Class of intensity models ...
  - Credit spreads only depend on the history of default events
    - Credit spreads are deterministic between two default dates
    - Default Risk governs Credit Spread Risk

- **Homogeneous credit portfolio**
  - No individual name effect
  - Only need of the CDS Index

- **Markovian dynamics of default intensities**
  - Pricing and hedging CDO within a binomial tree
Introduction

- Dynamic hedging of defaultable contingent claim in **complete market**
  - Blanchet-Scaillet & Jeanblanc [2004]
- Dynamic hedging of basket credit derivatives in **complete market**
  - Bielecki, Jeanblanc & Rutkowski [2007], Frey & Backhaus [2006]
- Dynamic hedging in **asymptotically complete market**
  - Laurent [2006]
- Dynamic hedging in **incomplete market**
  - Super-replication : Walker [2005]
  - Quadratic hedging : Becherer & Schweizer [2005], Elouerkhaoui [2006]
Martingale Representation Theorem

- Some notations:
  - $\tau_1, \ldots, \tau_n$: default dates of counterparties 1,...,n
  - $H_t$: natural filtration of default dates
  - $N_1(t) = 1_{\{\tau_1 \leq t\}}, \ldots, N_n(t) = 1_{\{\tau_n \leq t\}}$: default indicators at date t
  - $N(t) = \sum_{i=1}^{n} N_i(t)$: number of default at date t
  - $\alpha_1(t), \ldots, \alpha_n(t)$: spreads of instantaneous CDS
    - $\alpha_i(t)$ are default intensities of $N_1(t), \ldots, N_n(t)$
    - Probability $Q$ such that
      - under $Q$, $\alpha_1(t), \ldots, \alpha_n(t)$ are default intensities of $N_1(t), \ldots, N_n(t)$
Martingale Representation Theorem

- Integral representation of point process martingale
  - Jacod [1975], Brémaud Chap. III
  - **No simultaneous default**

\[
M = E^Q [M] + \sum_{i=1}^{n} \int_{0}^{T} \theta_i(s) \left( dN_i(s) - \alpha_i(s) ds \right)
\]

- \( M \): \( H_T \)-mesurable \( Q \)-integrable payoff
  - CDO Tranches payoff can be perfectly replicated
  - Using \( n \) instantaneous CDS

→ **Does not provide a practical way to construct hedging strategies**
Markovian homogeneous contagion model

- Contagion models: Davis & Lo[2001], Jarrow & Yu[2001], Yu[2001]
  - Default intensities depend on the complete history of defaults
    \[ Q\left(\tau_i \in \left[t,t+dt\right] \mid H_t \right) = \alpha_i(t, H_t) dt, \quad i = 1, \ldots, n \]

- Homogeneous assumption
  - Default intensities are the same for all names \( \alpha \)
  - Total loss is simply expressed as \( L(t) = (1 - R) \frac{N(t)}{n} \)

- Homogeneous + Markovian assumption
  - Default intensities only depend on the current number of defaults
    \[ Q\left(\tau_i \in \left[t,t+dt\right] \mid H_t \right) = Q\left(\tau_i \in \left[t,t+dt\right] \mid N_t \right) = \alpha(t, N(t)) dt, \quad i = 1, \ldots, n \]
Markovian homogeneous contagion model

- No simultaneous defaults assumption
  - Intensity $\lambda$ of the number of defaults process $N(t)$ is simply the sum of individual default intensities:

$$\lambda(t, N(t)) = (n - N(t)) \times \alpha(t, N(t))$$

- The process $N(t)$ is a Markov chain (a pure death process) with generator:

$$\Lambda(t) = \begin{bmatrix}
-\lambda(t,0) & \lambda(t,0) & 0 & 0 & 0 & 0 & 0 \\
0 & -\lambda(t,1) & \lambda(t,1) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

- $\{N(t) = n\}$ is an absorbing state
Tree Approach to hedging defaults

- Computation of Index and CDO tranche premiums
  - Based on the distribution of the aggregated loss $L(t) = (1 - R) \frac{N(t)}{n}$

- The transition matrix of $N(t)$ can be expressed as
  $$Q(t, t') = \exp\left(\int_t^{t'} \Lambda(s) ds\right)$$
  - Arnsdorf & Halperin[2007], Herbertsson[2007]

- Suppose that $k$ defaults have occurred at time $t$:
  $$Q\left(N(t + dt) = k + 1 \mid N(t) = k\right) \approx 1 - e^{-\lambda(t, k) dt}$$
  $$Q\left(N(t + dt) = k \mid N(t) = k\right) \approx e^{-\lambda(t, k) dt}$$
Tree Approach to hedging defaults

- Number of defaults tree (time homogeneous case)

- Calibration of $\lambda_0, \ldots, \lambda_n$ on marginal distribution of $N(t)$
  - forward induction

- Computation of CDO Tranches and Index present values
  - backward induction
Computation of deltas

- Calibration of loss intensities $\lambda_0, \ldots, \lambda_n$ on a gaussian copula distribution

- Homogeneous portfolio $n = 125$
- $T = 5$ years
- CDS Spreads: 20 bps per annum
- Recovery rate $R = 40\%$
- Correlation $\rho = 30\%$
- $Q(N(t) = k), \ k = 0, \ldots, 20$
Computation of deltas

- Calibration of loss intensities $\lambda_0, \ldots, \lambda_n$ on a gaussian copula distribution
  - Figure below represents loss intensities, with respect to the number of defaults
  - Increase in intensities: contagion effects
Computation of deltas

- **Dynamics of credit deltas**
  \[ \delta_{t,k} = \frac{CDO(t + 1, k + 1) - CDO(t + 1, k)}{Index(t + 1, k + 1) - Index(t + 1, k)} \]

- **Credit deltas - Tranche equity [0,3%]**

<table>
<thead>
<tr>
<th>Nb Defaults</th>
<th>OutStanding Nominal</th>
<th>0</th>
<th>14</th>
<th>28</th>
<th>42</th>
<th>56</th>
<th>70</th>
<th>84</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>3.00%</td>
<td>0.810</td>
<td>0.839</td>
<td>0.865</td>
<td>0.889</td>
<td>0.911</td>
<td>0.929</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2.52%</td>
<td>0</td>
<td>0.613</td>
<td>0.657</td>
<td>0.701</td>
<td>0.743</td>
<td>0.785</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2.04%</td>
<td>0</td>
<td>0.343</td>
<td>0.386</td>
<td>0.432</td>
<td>0.483</td>
<td>0.536</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.56%</td>
<td>0</td>
<td>0.142</td>
<td>0.167</td>
<td>0.197</td>
<td>0.231</td>
<td>0.271</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1.08%</td>
<td>0</td>
<td>0.046</td>
<td>0.055</td>
<td>0.066</td>
<td>0.080</td>
<td>0.097</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.60%</td>
<td>0</td>
<td>0.014</td>
<td>0.015</td>
<td>0.018</td>
<td>0.021</td>
<td>0.025</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.12%</td>
<td>0</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.003</td>
<td>0.003</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.00%</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- Gradually decrease with the number of defaults
  - concave payoff
  - When the number of default is > 6, the tranche is exhausted, delta = 0
- Credit deltas increase with time
## Computation of deltas

### Credit deltas - Tranche [3,6%]

<table>
<thead>
<tr>
<th>Nb Defaults</th>
<th>OutStanding Nominal</th>
<th>Weeks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>14</td>
</tr>
<tr>
<td>0</td>
<td>3.00%</td>
<td>0.162</td>
</tr>
<tr>
<td>1</td>
<td>3.00%</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3.00%</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3.00%</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>3.00%</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>3.00%</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>3.00%</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>2.64%</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>2.16%</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>1.68%</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>1.20%</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>0.72%</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>0.24%</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>0.00%</td>
<td>0</td>
</tr>
</tbody>
</table>

- When the number of default is > 12, the tranche is exhausted
Conclusion

- Thanks to stringent assumptions
  - Credit spreads driven by defaults
  - Homogeneity
  - Markov property
- It is possible to compute a dynamic hedging strategy
  - Based on the CDS Index
- That fully replicates the CDO tranche payoffs
  - Very simple implementation using a recombining tree
- Credit spread dynamics need to be improved