Hedging default risks of CDOs in Markovian contagion models

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Résumé :

Il s'agit d'une version concise de l'article "hedging default risks of CDOs in Markovian contation models" (2008) auquel nous renvoyons pour plus de dtails. Nous mettons en évidence une stratégie de duplication de tranches de CDO faisant appel au contrat de swap de défaut sur l'indice sous-jacent. La perte agrégée suit une chaîne de Markov. L'intensité de la perte agrégée dépend du nombre de défauts dans le portefeuille sous-jacent et il existe des phénomènes de contagion entre les entités constituant le portefeuille.

Abstract :

This contribution is an abridged version of the research paper "hedging default risks of CDOs in Markovian contagion models" (2008) to which we refer for further reading. We exhibit a replicating strategy of CDO tranches based upon dynamic trading of the corresponding credit default swap index. The aggregate loss follows a homogeneous Markov chain associated with contagion effects and default intensities depend upon the number of defaults.

Keywords: CDOs, hedging, complete markets, contagion model, Markov chain.

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Introduction

When dealing with CDO tranches, the market approach to the derivation of credit default swap deltas consists in bumping the credit curves of the names and computing the ratios of changes in present value of the CDO tranches and the hedging credit default swaps. This involves a pricing engine for CDO tranches, usually some mixture of copula and base correlation approaches, leading to some "market delta". The only rationale of this modus operandi is local hedging with respect to credit spread risks, provided that the trading books are marked-to-market with the same pricing engine. Even when dealing with small changes in credit spreads, there is no guarantee that this would lead to appropriate hedging strategies, especially to cover large spread widenings and possibly defaults. For instance one can think of changes in base correlation correlated with changes in credit spreads. A number of CDO hedging anomalies in the base correlation approach are reported in Morgan and Mortensen (2007). Moreover, the standard approach is not associated with a replicating theory, thus inducing the possibility of unexplained drifts and time decay effects in the present value of hedged portfolios (see Petrelli et al. (2007)).

Unfortunately, the trading desks cannot rely on a sound theory to determine replicating prices of CDO tranches. This is partly due to the dimensionality issue, partly to the stacking of credit spread and default risks. Laurent (2006) considers the case of multivariate intensities in a conditionally independent framework and shows that for large portfolios where default risks are well diversified, one can concentrate on the hedging of credit spread risks and control the hedging errors. In this approach, the key assumption is the absence of contagion effects which implies that credit spreads of survival names do not jump at default times, or equivalently that defaults are not informative. Whether one should rely on this assumption is to be considered with caution as discussed in Das et al. (2007). Anecdotal evidence such as the failures of Delphi, Enron, Parmalat and WorldCom shows mixed results.

In this paper, we take an alternative route, concentrating on default risks, credit spreads and dependence dynamics being driven by the arrival of defaults. We will calculate so-called "credit deltas", that are the present value impacts of some default event on a given CDO tranche, divided by the present value impact of the hedging instrument (here the underlying index) under the same scenario. Contagion models were introduced to the credit field by Davis and Lo (2001), Jarrow and Yu (2001) and further studied by Yu (2007). Schönbucher and Schubert (2001) show that copula models exhibit some contagion effects and relate jumps of credit spreads at default times to the partial derivatives of the copula. This is also the framework used by Bielecki et al. (2007b) to address the hedging issue. A similar but somehow more tractable approach has been considered by Frey and Backhaus (2007b), since the latter paper considers some Markovian models of contagion. In a copula model, the contagion effects are computed from the dependence structure of default times, while in contagion models the intensity dynamics are the inputs from which the dependence structure of default times is derived. In both approaches, credit spreads shifts occur only at default times. Thanks to this quite simplistic assumption, and provided that no simultaneous defaults occurs, it can be shown that the CDO market is complete, i.e. CDO tranche cash-flows can be fully replicated by dynamically trading individual credit spread swaps or, in some cases, by trading the credit default swap index.

Lately, Frey and Backhaus (2007a) have considered the hedging of CDO tranches in a Markov chain credit risk model allowing for spread and contagion risk. In this framework, when the hedging instruments are credit default swaps with a given maturity, the market is incomplete. In order to derive dynamic hedging strategies, Frey and Backhaus (2007a) use risk minimization techniques. In a multivariate Poisson model, Elouerkhaoui (2006) also addresses the hedging problem thanks to the risk minimization approach. As can be seen from the previous papers, practical implementation can be cumbersome, especially when dealing with the hedging ratios at different points in time and different states.

For the paper to be self-contained, we recall in Section 1 the mathematics behind the perfect replicating strategy. The main tool there is a martingale representation theorem for multivariate point processes. In Section 2, we restrict ourselves to the case of homogeneous portfolios with Markovian intensities which results in a dramatic dimensionality reduction for the (risk-neutral) valuation of CDO tranches and the hedging of such tranches as well. We find out that the aggregate loss is associated with a pure birth process, which is now well documented in the credit literature. Further details regarding the implementation of the model and numerical results are detailed in the comprehensive version of the paper.

1 Theoretical Framework

1.1 Default times

Throughout the paper, we will consider n obligors and a random vector of default times $(\tau_1, \ldots, \tau_n)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We denote by $N_1(t) = 1_{\{\tau_1 \le t\}}$, ..., $N_n(t) = 1_{\{\tau_n \le t\}}$, the default indicator processes and by $\mathcal{H}_{i,t} = \sigma\left(N_i(s), s \le t\right)$, $i = 1, \ldots, n$. $\mathcal{H}_t = \bigvee_{i=1}^{n} \mathcal{H}_{i,t}$. $(\mathcal{H}_t)_{t \in \mathbb{R}^+}$ is the natural filtration associated with the default times.

We denote by $\tau^1, \ldots, \tau^n$ the ordered default times and assume that no simultaneous defaults can occur, i.e. $\tau^1 < \ldots < \tau^n$, $\mathbb{P} - a.s.$ This assumption is important with respect to the completeness of the market. As shown below, it allows to dynamically hedge basket default swaps and CDOs with n credit default swaps[1].

We moreover assume that there exist some $(\mathbb{P}, \mathcal{H}_t)$ intensities for the default indicator processes $N_i(t)$, $i = 1, \ldots, n$, i.e. there exist some (non negative) $\mathcal{H}_t$–predictable processes $\alpha_i^{\mathbb{P}}$, $i = 1, \ldots, n$, such that:

$$N_i(t) - \int_0^t \alpha_i^{\mathbb{P}}(s) 1_{\{\tau_i > s\}} ds, \quad i = 1, \ldots, n,$$

are $(\mathbb{P}, \mathcal{H}_t)$–martingales. We moreover assume that for each name $i = 1, \ldots, n$, the corresponding default intensity $\alpha_i^{\mathbb{P}}$ vanishes after τ_i, i.e $\alpha_i^{\mathbb{P}}(t) = 0$ on the set $\{t > \tau_i\}$.

1.2 Market assumptions

For the sake of simplicity, let us assume for a while that instantaneous digital default swaps are traded on the names. An instantaneous digital credit default swap on name i traded at t, provides a payoff equal to $dN_i(t) - \alpha_i(t)dt$ at $t + dt$. $dN_i(t)$ is the payment on the default leg and $\alpha_i(t)dt$ is the (short term) premium on the default swap. As there are no more cash-flows after default of name i, $\alpha_i(t) = 0$ on the set $\{t > \tau_i\}$. Note that considering such instantaneous digital default swaps rather than actually traded credit default swaps is

[1] In the general case where multiple defaults could occur, we have to consider possibly 2^n states, and we would require non standard credit default swaps with default payments conditionally on all sets of multiple defaults to hedge CDO tranches.

not a limitation of our purpose. This can rather be seen as a convenient choice of basis from a theoretical point of view. Of course, we will compute credit deltas with respect to traded credit default swaps in the applications below[2].

Since we deal with the filtration generated by default times, the credit default swap premiums are deterministic between two default events. Therefore, we restrain ourselves to a market where only default risks occurs and credit spreads themselves are driven by the occurrence of defaults. In our simple setting, there is no specific credit spread risk. This corresponds to the framework of Bielecki et al. (2007a).

For simplicity, we further assume that (continuously compounded) default-free interest rates are constant and equal to r. Given some initial investment V_0 and some $\mathcal{H}_t$–predictable processes $\delta_1(.), \ldots, \delta_n(.)$ associated with some self-financed trading strategy in instantaneous digital credit default swaps, we attain at time T the payoff:

$$V_0 e^{rT} + \sum_{i=1}^{n} \int_0^T \delta_i\left(s\right) e^{r(T-s)} \left(dN_i(s) - \alpha_i(s)ds\right).$$

By definition, $\delta_i(s)$ is the nominal amount of instantaneous digital credit default swap on name i held at time s. This induces a net cash-flow of $\delta_i(s) \times (dN_i(s) - \alpha_i(s)ds)$ at time $s + ds$, which has to be invested in the default-free savings account up to time T.

1.3 Hedging and martingale representation theorem

From the absence of arbitrage opportunities, $\alpha_1, \ldots, \alpha_n$ are non negative $\mathcal{H}_t$–predictable processes. From the same reason, $\{\alpha_i(t) > 0\} \overset{\mathbb{P}-a.s.}{=} \left\{\alpha_i^{\mathbb{P}}(t) > 0\right\}$. Under mild regularity assumptions, there exists a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the instantaneous credit default swap premiums $\alpha_1, \ldots, \alpha_n$ are the $(\mathbb{Q}, \mathcal{H}_t)$–intensities associated with the default times (see Brémaud (1981), chapter VI). Therefore, from now on, the premiums will be denoted $\alpha_1^{\mathbb{Q}}, \ldots, \alpha_n^{\mathbb{Q}}$ and we will work under the probability $\mathbb{Q}$.

Let us consider some $\mathcal{H}_T$–measurable $\mathbb{Q}$–integrable payoff M. Since M depends upon the default indicators of the names up to time T, this encompasses the cases of CDO tranches and basket default swaps, provided that recovery rates are deterministic. Thanks to the integral representation theorem of point process martingales (see Brémaud (1981), chapter III), there exists some $\mathcal{H}_t$–predictable processes $\theta_1, \ldots, \theta_n$ such that:

$$M = \mathbb{E}^{\mathbb{Q}}\left[M\right] + \sum_{i=1}^{n} \int_0^T \theta_i(s) \left(dN_i(s) - \alpha_i^{\mathbb{Q}}(s)ds\right).$$

As a consequence, we can replicate M with the initial investment $\mathbb{E}^{\mathbb{Q}}\left[Me^{-rT}\right]$ and the trading strategy based on instantaneous digital credit default swaps defined by $\delta_i(s) = \theta_i(s)e^{-r(T-s)}$ for $0 \leq s \leq T$ and $i = 1, \ldots, n$. Let us remark that the replication price at time t, is provided by $V_t = \mathbb{E}^{\mathbb{Q}}\left[Me^{-r(T-t)} \left|\mathcal{H}_t\right.\right]$[3].

[2]Note that the instantaneous credit default swaps are not exposed to spread risk but only to default risk.

[3]Let us notice that $M = \mathbb{E}^{\mathbb{Q}}\left[M \left|\mathcal{H}_t\right.\right] + \sum_{i=1}^{n} \int_t^T \theta_i(s) \left(dN_i(s) - \alpha_i^{\mathbb{Q}}(s)ds\right)$. As a consequence, we readily get $M = V_t e^{r(T-t)} + \sum_{i=1}^{n} \int_t^T \theta_i(s) \left(dN_i(s) - \alpha_i^{\mathbb{Q}}(s)ds\right)$ which provides the time t replication price of M. We can

While the use of the representation theorem guarantees that, in our framework, any basket default swap can be perfectly hedged with respect to default risks, it does not provide a practical way of constructing hedging strategies. As is the case with interest rate or equity derivatives, exhibiting hedging strategies involves some Markovian assumptions (see Subsection 2.3).

2 Homogeneous Markovian contagion models

2.1 Intensity specification

In the contagion approach, one starts from a specification of the risk-neutral pre-default intensities $\alpha_1^{\mathbb{Q}}, \ldots, \alpha_n^{\mathbb{Q}}$[4]. In the previous section framework, the risk-neutral default intensities depend upon the complete history of defaults. More simplistically, it is often assumed that they depend only upon the current credit status, i.e. the default indicators; thus $\alpha_i^{\mathbb{Q}}(t)$, $i = 1, \ldots, n$ are deterministic functions of $N_1(t), \ldots, N_n(t)$. In this paper, we will further remain in this Markovian framework, i.e. the pre-default intensities will take the form $\alpha_i^{\mathbb{Q}}(t, N_1(t), \ldots, N_n(t))$[5]. Popular examples are the models of Kusuoka (1999), Jarrow and Yu (2001), Yu (2007), where the intensities are affine functions of the default indicators. The connection between contagion models and Markov chains is described in the book of Lando (2004) and was further discussed in Herbertsson (2007).

Another practical issue is related to name heterogeneity. Modelling all possible interactions amongst names leads to a huge number of contagion parameters and high dimensional problems, thus to numerical issues. For this practical purpose, we will further restrict to models where all the names share the same risk-neutral intensity[6]. This can be viewed as a reasonable assumption for CDO tranches on large indices, although this is obviously an issue with equity tranches for which idiosyncratic risk is an important feature. Since pre-default risk-neutral default intensities, $\alpha_1^{\mathbb{Q}}, \ldots, \alpha_n^{\mathbb{Q}}$ are equal, we will further denote these individual pre-default intensities by $\alpha^{\mathbb{Q}}$.

For further tractability, we will further rely on a strong name homogeneity assumption, that individual pre-default intensities only depend upon the number of defaults. Let us denote by $N(t) = \sum_{i=1}^{n} N_i(t)$ the number of defaults at time t within the pool of assets. Pre-default intensities thus take the form $\alpha^{\mathbb{Q}}(t, N(t))$[7]. This is related to mean-field approaches (see Frey and Backhaus (2007b)). As for parametric specifications, we can think of some additive effects, i.e. the pre-default name intensities take the form $\alpha^{\mathbb{Q}}(t) = \alpha + \beta N(t)$ for some constants α, β as mentioned in Frey and Backhaus (2007b), corresponding to the "lin-

also remark that for a small time interval dt, $V_{t+dt} \approx V_t(1+r)dt + \sum_{i=1}^{n} \delta_i(t)\left(dN_i(t) - \alpha_i^{\mathbb{Q}}(t)dt\right)$ which is consistent with market practice and regular rebalancing of the replicating portfolio. An investor who wants to be compensated at time t against the price fluctuations of M during a small period dt has to invest V_t in the risk-free asset and take positions $\delta_1, \ldots, \delta_n$ in the n instantaneous digital credit default swaps. Let us recall that there is no initial charge to enter in a credit default swap position.

[4]After default of name i, the intensity is equal to zero: $\alpha_i^{\mathbb{Q}}(t) = 0$ on the set $\{t > \tau_i\}$.

[5]This Markovian assumption may be questionable, since the contagion effect of a default event may vanish as time goes by. The Hawkes process, that was used in the credit field by Giesecke and Goldberg (2006), Errais et al. (2007), provides such an example of a more complex time dependence. Other specifications with the same aim are discussed in Lopatin and Misirpashaev (2007).

[6]This means that the pre-default intensities have the same functional dependence to the default indicators.

[7]Let us remark that on $\{\tau_i \geq t\}$, $N(t) = \sum_{j \neq i} N_j(t)$, so that the pre-default intensity of name i, actually only depends on the credit status of the other names.

ear counterparty risk model"[8], or multiplicative effects in the spirit of Davis and Lo (2001), i.e. the pre-default intensities take the form $\alpha^{\mathbb{Q}}(t) = \alpha \times \beta^{N(t)}$. Of course, we could think of a non-parametric model[9].

For simplicity, we will further assume a constant recovery rate equal to R and a constant exposure among the underlying names. The aggregate fractional loss at time t is given by: $L(t) = (1-R)\frac{N(t)}{n}$. As a consequence of the no simultaneous defaults assumption, the intensity of $L(t)$ or of $N(t)$ is simply the sum of the individual default intensities and is itself only a function of the number of defaults process. Let us denote by $\lambda(t, N(t))$ the risk-neutral loss intensity. It is related to the individual pre-default risk-intensities by:

$$\lambda(t, N(t)) = (n - N(t)) \times \alpha^{\mathbb{Q}}(t, N(t)).$$

We are thus typically in a bottom-up approach, where one starts with the specification of name intensities and thus derives the dynamics of the aggregate loss.

2.2 Risk-neutral pricing

Let us remark that in a Markovian homogeneous contagion model, the process $N(t)$ is a Markov chain (under the risk-neutral probability $\mathbb{Q}$), and more precisely a pure birth process, according to Karlin and Taylor (1975) terminology[10], since only single defaults can occur[11]. The generator of the chain, $\Lambda(t)$ is quite simple:

$$\Lambda(t) = \begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & & 0 \\ & & \ddots & \ddots & \\ 0 & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Such a simple model of the number of defaults dynamics was considered by Schönbucher (2006) where it is called the "one-step representation of the loss distribution". Our paper can be seen as a bottom-up view of the previous model, where the risk-neutral prices can actually be viewed as replicating prices. As an example of this approach, let us consider the replication price of a European payoff with payment date T, such as a "zero-coupon tranchelet", paying $1_{\{N(T)=k\}}$ at time T for some $k \in \{0, 1, \ldots, n\}$. Let us denote by $V(t, N(t)) = e^{-r(T-t)} Q(N(T) = k \,|N(t))$ the time t replication price and by $V(t, .)$ the price vector whose components are $V(t,0), V(t,1), \ldots, V(t,n)$ for $0 \le t \le T$. We can thus relate the price vector $V(t, .)$ to the terminal payoff, using the transition matrix $Q(t,T)$ between dates t and T:

$$V(t, .) = e^{-r(T-t)} Q(t,T) V(T, .),$$

where $V(T, N(T)) = \delta_k(N(T))$. The transition matrix solves for the Kolmogorov backward and forward equations $\frac{\partial Q(t,T)}{\partial t} = -\Lambda(t) Q(t,T)$, $\frac{\partial Q(t,T)}{\partial T} = Q(t,T)\Lambda(T)$. In the time homogeneous case, i.e. when the generator is a constant $\Lambda(t) = \Lambda$, the transition matrix can be

[8] Ding et al. (2006) consider the case where the intensity of the loss process is linear in the number of defaults. Then, the loss distribution is negative binomial.

[9] We provide a calibration procedure of such unconstrained intensities onto market inputs in the comprehensive version of the paper.

[10] According to Feller's terminology, we should speak of a pure death process. Since, we later refer to Karlin and Taylor (1975), we prefer their terminology.

[11] Regarding the assumption of no simultaneous defaults, we also refer to Putyatin et al. (2005), Brigo et al. (2007), Walker (2007). Allowing for multiple defaults could actually ease the calibration onto senior CDO tranche quotes.

written in exponential form $Q(t,T) = \exp\left((T-t)\Lambda\right)$.

These ideas have been put in practice by van der Voort (2006), Herbertsson and Rootz'en (2006), Arnsdorf and Halperin (2007), De Koch and Kraft (2007), Epple et al. (2007), Herbertsson (2007) and Lopatin and Misirpashaev (2007). These papers focus on the pricing of credit derivatives, while our concern here is the feasibility and implementation of replicating strategies.

2.3 Computation of credit deltas

We recall that the credit delta with respect to name i is the amount of hedging instruments (the index here, but possibly a i-th credit default swap) that should be bought to be protected against a sudden default of name i. A nice feature of homogeneous contagion models is that the credit deltas are the same for all (the non-defaulted) names, which results in a dramatic dimensionality reduction.

Let us consider a European type payoff[12] and denote its replication price at time t by $V(t,.)$. In order to compute the credit deltas, let us remark that, by Ito's lemma,

$$dV\left(t,N(t)\right) = \frac{\partial V\left(t,N(t)\right)}{\partial t}dt + \left(V\left(t,N(t)+1\right) - V\left(t,N(t)\right)\right)dN(t).$$

$V\left(t,N(t)+1\right) - V\left(t,N(t)\right)$ is associated with the jump in the price process when a default occurs in the credit portfolio, i.e. $dN(t) = 1$. Thanks to the name homogeneity, $dN(t) = \sum_{i=1}^{n-N(t)} dN_i(t)$[13] and, since $(e^{-r(T-t)}V(t,N(t)))$ is a $\mathbb{Q}$–martingale,

$$\frac{\partial V\left(t,N(t)\right)}{\partial t} + \lambda\left(t,N(t)\right) \times \left(V\left(t,N(t)+1\right) - V\left(t,N(t)\right)\right) = rV\left(t,N(t)\right),$$

we end up with:

$$\begin{aligned} dV\left(t,N(t)\right) &= rV\left(t,N(t)\right)dt \\ &\quad + \sum_{i=1}^{n}\left(V(t,N(t)+1) - V(t,N(t))\right)\left(dN_i(t) - \alpha^{\mathbb{Q}}(t,N(t))dt\right). \end{aligned}$$

As a consequence the credit deltas with respect to the individual instantaneous default swaps are equal to:

$$\delta_i(t) = e^{-r(T-t)}\left(V\left(t,N(t)+1\right) - V\left(t,N(t)\right)\right) \times \left(1 - N_i(t)\right),$$

for $0 \leq t \leq T$ and $i = 1,\ldots,n$.
Let us denote by $V_I(t,k) = e^{-r(T-t)}E^Q\left[1 - \frac{N(T)}{n}|N(t) = k\right]$ the time t price of the equally weighted portfolio involving defaultable discount bonds and set

$$\delta_I\left(t,N(t)\right) = \frac{V\left(t,N(t)+1\right) - V\left(t,N(t)\right)}{V_I\left(t,N(t)+1\right) - V_I\left(t,N(t)\right)}.$$

[12]For notational simplicity, we assume that there are no intermediate payments. This corresponds for instance to the case of zero-coupon CDO tranches with up-front premiums. The more general case is considered in the comprehensive version of the paper.

[13]The last $N(t)$ names have defaulted.

It can readily be seen that:

$$dV\left(t,N(t)\right)=r\times\left(V\left(t,N(t)\right)-\delta_I\left(t,N(t)\right)V_I\left(t,N(t)\right)\right)dt+\delta_I\left(t,N(t)\right)dV_I\left(t,N(t)\right).$$

As a consequence, we can perfectly hedge a European type payoff, say a zero-coupon CDO tranche, using only the index portfolio and the risk-free asset. The hedge ratio, with respect to the index portfolio is actually equal to

$$\delta_I\left(t,N(t)\right)=\frac{V\left(t,N(t)+1\right)-V\left(t,N(t)\right)}{V_I\left(t,N(t)+1\right)-V_I\left(t,N(t)\right)}.$$

The previous hedging strategy is feasible provided that $V_I\left(t,N(t)+1\right)\neq V_I\left(t,N(t)\right)$. The usual case corresponds to some positive dependence, thus $\alpha^{\mathbb{Q}}\left(t,0\right)\leq\alpha^{\mathbb{Q}}\left(t,1\right)\leq\cdots\leq\alpha^{\mathbb{Q}}\left(t,n-1\right)$. Therefore $V_I\left(t,N(t)+1\right)<V_I\left(t,N(t)\right)$[14]. The decrease in the index portfolio value is the consequence of a direct default effect (one name defaults) and an indirect effect related to a positive shift in the credit spreads associated with the non-defaulted names.

The idea of building a hedging strategy based on the change in value at default times was introduced in Arvanitis and Laurent (1999). The rigorous construction of a dynamic hedging strategy in a univariate case can be found in Blanchet-Scalliet and Jeanblanc (2004). Our result can be seen as a natural extension to the multivariate case, provided that we deal with Markovian homogeneous models: we simply need to deal with the number of defaults $N(t)$ and the index portfolio $V_I\left(t,N(t)\right)$ instead of a single default indicator $N_i(t)$ and the corresponding defaultable discount bond price.

Conclusion

The lack of internally consistent methods to hedge CDO tranches has paved the way to a variety of local hedging approaches that do not guarantee the full replication of tranche payoffs. This may not look as such a practical issue when trade margins are high and holding periods short. However, we think that there might be a growing concern from investment banks about the long term credit risk management of trading books as the market matures.

A homogeneous Markovian contagion model provides a strikingly easy way to compute dynamic replicating strategies of CDO tranches. While such models have recently been considered for the pricing of exotic basket credit derivatives, our main concern here is to provide a rigorous framework to the hedging issue.

We do not aim at providing a definitive answer to the thorny issue of hedging CDO tranches. For this purpose, we would also need to tackle name heterogeneity, possible non Markovian effects in the dynamics of credit spreads, non deterministic intensities between two default dates, the occurrence of multiple defaults, stochastic recovery rates... A fully comprehensive approach to the hedging of CDO tranches is likely to be quite cumbersome both on economic and numerical grounds.

However, from a practical perspective, we think that our approach might be useful to assess the default exposure of CDO tranches by quantifying the credit contagion effects in a reasonable way.

[14] In the case where $\alpha^{\mathbb{Q}}(t,0)=\alpha^{\mathbb{Q}}(t,1)=\ldots=\alpha^{\mathbb{Q}}(t,n)$, there are no contagion effects and default dates are independent. We still have $V_I(t,N(t)+1)<V_I(t,N(t))$ since $V_I(t,N(t))$ is linear in the number of surviving names.

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