Comparison analysis of two alternative hedging methods for CDO tranches

Areski COUSIN

Université d'Evry Val d'Essonne

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Joint work with Stéphane Crépey

Introduction and motivation

- Recent financial turmoil has deeply affected the market of structured credit derivatives
- CDS index products are still liquid but ...
- Investors have more incentive to risk-manage their trading books
- However, standard hedging methods have revealed serious drawbacks during the crisis
  - focus on the computation of spread sensitivities (credit deltas)
  - within a static Gaussian copula model
  - does not rely on a sound theory of replication
  - negative deltas may occur in a steep base correlation market: Morgan and Mortensen (2007)

Introduction and motivation

- We consider discrete-time hedging of index CDO tranches using
  - the underlying CDS index
  - the risk-free asset
- Hedging consists in taking complementary positions in the index and in the risk-free asset in order to minimize the overall evolution of market prices.
  - These positions need to be regularly updated over time
- **Aim of the presentation**: performance analysis of two alternative hedging strategies
  - Δ^{lo}: delta of the tranche within a Markovian contagion model
  - Δ^{li}: delta of the tranche within a Gaussian copula model

In the literature

- Crépey (2004) performs a similar analysis for the equity market
  - Comparison of hedging performance of equity options using two alternative deltas: Black-Scholes implied delta and local volatility delta
  - He exhibits two market directions: (slow/fast) and (rallies/sell-offs)
  - Negatively skew market: local volatility delta provides a better hedge than implied delta during slow rallies or fast sell-offs and a worse hedge during fast rallies and slow sell-offs.
- Analogies with the credit market are not so obvious
  - Interaction between default risk and spread risk, large dimension of the portfolio, recovery rate uncertainty

In the literature

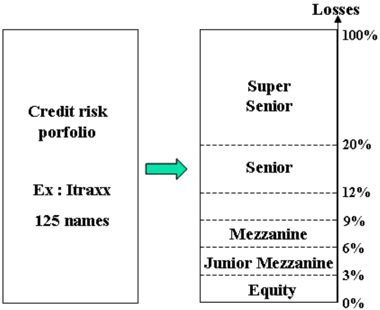
- Laurent, Cousin and Fermanian (2007) study the hedging of index CDO tranche in a Markovian contagion model using the CDS index
  - When simultaneous defaults are precluded, the CDO tranche market is complete
  - Computation of dynamic hedging strategies along the nodes of a binomial tree
  - Δ^{lo} seems to be smaller than Δ^{li} for equity tranches and higher for more senior tranches when the two models are calibrated to the same market data.
  - Do not study in details the performance of these two hedging strategies

In the literature

- Cont and Kan (2008) perform an empirical comparison of various hedging strategies for index CDO tranches
  - three different notions of deltas: spread-delta, jump-to-default delta, quadratic risk-minimization delta
  - deltas computed in various models calibrated to the same set of market data
  - Backtest the strategies before and during the crisis
- Main conclusions:
  - spread-deltas are very similar across models calibrated to the same data set
  - jump-to-default ratios are significatively different across models (substantial model risk)
  - spread-deltas hedges are preferred before the crisis and not preferred during the crisis
- But study of performance only for a single (market) trajectory
- Do not address the issue of hedging with individual CDS

CDS Index and standardized CDO tranches

- Slice the credit portfolio into different risk levels or CDO tranches
- ex: CDO tranche on standardized Index such as CDX North America or Itraxx Europe



Notations

- Credit portfolio with n reference entities
- $\tau_1, \ldots, \tau_n$: default times
- R: homogeneous and constant recovery rate at default ($R = 40\%$ typically)
- Number of defaults process:

$$N_t = \sum_{i=1}^{n} 1_{\{\tau_i \leq t\}}$$

- CDO tranche cash-flows are driven by the aggregate loss process normalized to unity:

$$L_t = \frac{1}{n}(1 - R)N_t$$

- Cash-flows only depends on $\phi(L_t)$, $0 \leq t \leq T$

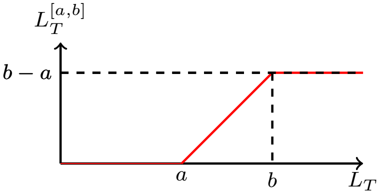
Assumptions

- By simplicity, we consider zero interest rate $r = 0$
- Stylized products with simplified cash-flows:
  - Cash-flows of the index and associated CDO tranches only depend on $\phi(L_T)$
  - Protection or default payment only occur at maturity T
  - The premium leg is paid upfront
- Given a risk-neutral probability $\mathbb{P}$ and a fitration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$
  - The time t price of a derivative with a $\mathcal{F}_T$-measurable and bounded payoff $\xi = \phi(L_T)$ is:

$$\mathbb{E}[\xi \mid \mathcal{F}_t]$$

CDS Index and standardized CDO tranches

- Loss on CDO tranche $[a, b]$: $L_T^{[a,b]}$ has a call spread payoff with respect to L_T:



- Loss on CDO tranche $[a, b]$:
  $L_T^{[a,b]} = (L_T - a)^+ - (L_T - b)^+ = \phi^{[a,b]}(N_T)$
  - equity tranches: $a = 0\%$ and $L_t^{[0,b]} = \min(L_T, b)$
  - senior tranches: $b = 100\%$ and $L_t^{[a,1]} = (L_T - a)^+$
  - CDS index associated with a $[0\%, 100\%]$ CDO tranche

CDS Index and standardized CDO tranches

- As the premium leg is paid upfront, the analysis is focused on the **protection leg**
- The time-t cum-dividend price of a stylized CDO tranche $[a, b]$ (protection leg) is referred to as:

$$\Pi_t = \mathbb{E}\left[\phi^{[a,b]}(N_T) \mid \mathcal{F}_t\right]$$

- The time-t cum-dividend price of the stylized underlying index (protection leg) is referred to as:

$$P_t = \mathbb{E}\left[\phi^{[0,1]}(N_T) \mid \mathcal{F}_t\right]$$

Homogeneous one factor Gaussian copula model

- Also referred to as the **Li model**
- $V_i = \rho V + \sqrt{1-\rho^2}\bar{V}_i,\ i = 1 \dots n$: latent variables
    - $V, \bar{V}_i,\ i = 1 \dots n$: independent Gaussian random variables
- Default times defined by: $\tau_i = F_i^{-1}(\Phi(V_i)),\ i = 1 \dots n$
    - $F_1 = \dots = F_n = F$: cdf of τ_i, $i = 1, \dots, n$
    - Φ: cdf of V_i
- Conditional default probability

$$p_t(V) = \mathbb{P}(\tau_i \le t \mid V) = \Phi\left(\frac{\Phi^{-1}(F(t)) - \rho V}{\sqrt{1-\rho^2}}\right)$$

- Loss distribution is merely a binomial mixture:

$$\mathbb{P}(N_t = k) = \binom{n}{k} \int p_t(x)^k (1 - p_t(x))^{n-k} \nu(x) dx, \quad k = 0, \dots, n$$

Homogeneous one factor Gaussian copula model

- At each time t, the model parameters ρ_t and F_t are calibrated on market spreads
- F_t is inferred from the term structure of index spreads at time t
  - Index spread curve assumed to be flat and equal to S_t

$$F_t(s) = \mathbb{P}(\tau_i \leq t) = 1 - \exp\left(-\frac{S_t}{1-R}(s-t)\right), \; s \geq t$$

- One dependence parameter ρ_t^b associated with each base tranche $[0, b]$, $b =$ 3%, 6%, 9%, 12%, 22% (iTraxx)
  - $\Pi_t^{ma}(T, a, b)$: market price of CDO tranche $[a, b]$, maturity T
  - $\Pi^{li}(T, a, b; t, S_t, \rho_t)$: price of CDO tranche $[a, b]$ in the Li model
  - Base correlation ρ_t^b is such that:

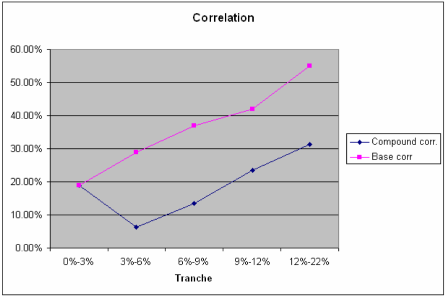
$$\Pi^{li}(T, 0, b; t, S_t, \rho_t^b) = \Pi_t^{ma}(T, 0, b)$$

Homogeneous one factor Gaussian copula model

- Monotonic base tranche and senior tranche prices with respect to ρ in the Li model

$$\frac{\partial \Pi^{li}(T,0,b;\, t, S_t, \rho)}{\partial \rho} \leq 0, \quad \frac{\partial \Pi^{li}(T,a,1;\, t, S_t, \rho)}{\partial \rho} \geq 0, \quad \forall a, b \in [0,1]$$

  - Given the existence of base correlations ρ_t^b, these parameters are unique
- CDO tranche market typically reflects steep base correlation curves:



Homogeneous Markovian contagion model

- Also referred to as the **local intensity model**
- Dynamic model where the cumulative default intensity only depends on number of defaults
- N_t is a continuous-time Markov chain with generator matrix:

$$\Lambda(t) = \begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & & 0 \\ & & \ddots & \ddots & \\ 0 & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- $\Pi^{lo}(t,T) = \mathbb{E}\left[\Phi(N_T) \mid \mathcal{F}_t\right] = \mathbb{E}\left[\Phi(N_T) \mid N_t\right]$

Homogeneous Markovian contagion model

- Vector of prices $\Pi^{lo}(t,T) = \left(\Pi_1^{lo}(t,T), \dots, \Pi_n^{lo}(t,T)\right)^\top$
  - where $\Pi_i^{lo}(t,T) = \mathbb{E}\left[\Phi(N_T) \mid N_t = i\right]$, $i = 1, \dots, n$
- can be related to the vector of terminal payoffs $C = (\Phi(0), \dots, \Phi(n))^\top$
- using the backward Kolmogorov equation:

$$\begin{cases} \frac{\partial \Pi^{lo}(t,T)}{\partial t} = -\Lambda(t)\Pi^{lo}(t,T) \\ \Pi^{lo}(T,T) = C \end{cases}$$

- When the intensities are time-homogeneous, i.e, $\Lambda(t) = \Lambda$ then:

$$\Pi^{lo}(t,T) = \exp\left((T-t)\Lambda\right) C$$

- Approach puts in practice by van der Voort (2006), Herbersson (2007), Laurent, Cousin and Fermanian (2007), Arnsdorf and Halperin (2007), Lopatin and Misirpashaev (2007), Cont and Minca (2008)

Delta-hedging in discrete time

- An investor enters a sell-protection position on a CDO tranche $[a, b]$
- He wants to cover his position until an hedging horizon: $T_1 \leq T$
- **Delta-hedging** the tranche consists in rebalancing a complementary position in a portfolio including the underlying index and the risk-free asset
  - at every point in time of a subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_p = T_1$
  - index position determined in order to minimize the overall exposure

Delta-hedging in discrete time

- The profit-and-loss (P&L) trajectory $e = (e_{t_k})_{0\leq k\leq p}$ is obtained by adding up the P&L increments:

$$\delta_k e = -\delta_k \Pi + \Delta_{t_k} \delta_k P$$

  - $\delta_k \Pi = \Pi_{t_{k+1}} - \Pi_{t_k}$: increments of the tranche market price in $(t_k, t_{k+1}]$
  - $\delta_k P = P_{t_{k+1}} - P_{t_k}$: increments of the index market price in $(t_k, t_{k+1}]$
  - Δ_{t_k}: number of units of index contract in the hedging portfolio over the time interval $(t_k, t_{k+1}]$
- Aim is to compare the $P\&L$ trajectory e obtained using two strategies:
  - $\Delta = \Delta^{lo}$: delta of the tranche in a Markovian contagion model
  - $\Delta = \Delta^{li}$: delta of the tranche in a Gaussian copula model

Delta-hedging in discrete time

- Δ^{lo}: jump-to-default in the local intensity model

$$\Delta_t^{lo} = \frac{\Pi_{i+1}^{lo}(t) - \Pi_i^{lo}(t)}{P_{i+1}^{lo}(t) - P_i^{lo}(t)}$$

  - where $\Pi_i^{lo}(t) = \mathbb{E}\left[\Phi^{[a,b]}(N_T) \mid N_t = i\right]$
  - and $P_i^{lo}(t) = \mathbb{E}\left[\Phi^{[0,1]}(N_T) \mid N_t = i\right]$

- Δ^{li}: spread-delta (sticky strike rule) in the Li model

$$\Delta_t^{li} = \frac{\Pi^{li}(t, S_t + \varepsilon, \rho_t) - \Pi^{li}(t, S_t, \rho_t)}{P^{li}(t, S_t + \varepsilon) - P^{li}(t, S_t)}$$

  - ε is typically equal to some few basis points

- The two models are calibrated on the same set of market spreads at every time t_k, $k = 0, \ldots, p-1$

Analysis in a Markovian contagion model

- Methodology similar to Hull and Suo (2000) and Crépey (2004)
- Theoretical market given as a fixed Markovian contagion model:
  - Market prices are such that:

$$\Pi_t = \Pi^{lo}(t, N_t), \; P_t = P^{lo}(t, N_t), \; S_t = S^{lo}(t, N_t), \; \rho_t = \rho^{lo}(t, N_t)$$

  - Given $N_t = i$, $\Pi_t = \Pi_i^{lo}(t)$, $P_t = P_i^{lo}(t)$, $S_t = S_i^{lo}(t)$, $\rho_t = \rho_i^{lo}(t)$
- As the CDO tranche market is complete in homogeneous Markovian contagion model, Δ^{lo} is the perfect continuous-time hedging strategy
- But we consider hedging in discrete time ...

Analysis in a Markovian contagion model

- Simulation of $\bar{N}$ default trajectories in the local intensity model
  - Simulation of $\left(\tau^{(1)}, \ldots, \tau^{(n)}\right)_j$, $j = 1, \ldots, \bar{N}$
- Without loss of generality, we focus the hedging analysis on a single period $(t_k, t_{k+1}]$
- $\Delta_{t_k}^{lo}$ is preferred to $\Delta_{t_k}^{li}$ on the period $(t_k, t_{k+1}]$ if

$$\text{Var}\left(\delta_k e^{lo}\right) < \text{Var}\left(\delta_k e^{li}\right)$$

  - where $\delta_k e^{lo}$ is the P&L increment in $(t_k, t_{k+1}]$ using $\Delta_{t_k}^{lo}$
  - where $\delta_k e^{li}$ is the P&L increment in $(t_k, t_{k+1}]$ using $\Delta_{t_k}^{li}$

Analysis in a Markovian contagion model

- As in Crépey (2004) we distinguish two market directions (slow/fast) and (rallies/sell-offs)
- Regarding the period $(t_k, t_{k+1}]$, a market trajectory is said to be
  - **fast**: a default is observed on the period $(t_k, t_{k+1}]$
  - **slow**: no default is observed on the period $(t_k, t_{k+1}]$
  - **rallies**: $\delta_k P \leq 0$ (decreasing index spread)
  - **sell-offs**: $\delta_k P \geq 0$ (increasing index spread)

Analysis in a Markovian contagion model

- We consider the hedging of an **equity tranche** (Analysis is similar for a senior tranche)

Proposition

$\delta_k e^{lo}$ is **positive** at **slow** market regimes and **negative** at **fast** market regimes

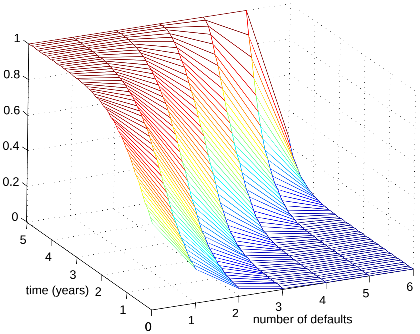
- Indeed, one can remark that:

$$\delta_k e^{lo} = -\delta_k \Pi + \Delta_{t_k}^{lo} \delta_k P = \int_{t_k}^{t_{k+1}} \left(\Delta_{t_k}^{lo} - \Delta_t^{lo} \right) dP_t$$

  - Consider a small interval $(t_k, t_{k+1}]$
  - no default (slow): $dP_t \simeq \delta_k P \leq 0$ (time decay effect)
  - one default (fast): $dP_t \simeq \delta_k P \geq 0$ (cash-flow and contagion effect)

Analysis in a Markovian contagion model

- And Δ_t^{lo} is typically increasing in t:



- $(t,i) \to \Delta^{lo}(t,i) = \frac{\Pi_{i+1}^{lo}(t) - \Pi_i^{lo}(t)}{P_{i+1}^{lo}(t) - P_i^{lo}(t)}$, $0 \leq t \leq 5$, $i = 0, \ldots, 6$

Analysis in a Markovian contagion model

- **Ordering of the two deltas** for equity tranche

Proposition

- If $\rho_{i+1}^{lo}(t) \geq \rho_i^{lo}(t)$, then one may expect that $\Delta_t^{lo} \leq \Delta_t^{li}$
- If $\rho_{i+1}^{lo}(t) \leq \rho_i^{lo}(t)$, then one may expect that $\Delta_t^{lo} \geq \Delta_t^{li}$

- Indeed, by definition of the implied base correlation:

$$\begin{aligned}
\Pi_{i+1}^{lo}(t) - \Pi_i^{lo}(t) &= \Pi^{li}\left(t, S_{i+1}^{lo}(t), \rho_{i+1}^{lo}(t)\right) - \Pi^{li}\left(t, S_i^{lo}(t), \rho_i^{lo}(t)\right) \\
&= \Pi^{li}\left(t, S_{i+1}^{lo}(t), \rho_{i+1}^{lo}(t)\right) - \Pi^{li}\left(t, S_{i+1}^{lo}(t), \rho_i^{lo}(t)\right) \\
&+ \Pi^{li}\left(t, S_{i+1}^{lo}(t), \rho_i^{lo}(t)\right) - \Pi^{li}\left(t, S_i^{lo}(t), \rho_i^{lo}(t)\right)
\end{aligned}$$

- But as $\partial_\rho \Pi^{li}(t, S, \rho) \leq 0$ for an equity tranche:

$$\Pi^{li}\left(t, S_{i+1}^{lo}(t), \rho_{i+1}^{lo}(t)\right) \leq \Pi^{li}\left(t, S_{i+1}^{lo}(t), \rho_i^{lo}(t)\right)$$

Analysis in a Markovian contagion model

- **Ordering of the two deltas** (cont.)
- By definition of the local intensity delta:

$$\begin{aligned}
\Delta^{lo}(t,i) &= \Delta^{lo}(t,i) = \frac{\Pi^{lo}_{i+1}(t) - \Pi^{lo}_{i}(t)}{P^{lo}_{i+1}(t) - P^{lo}_{i}(t)} \\
&\leq \frac{\Pi^{li}\left(t, S^{lo}_{i+1}(t), \rho^{lo}_{i}(t)\right) - \Pi^{li}\left(t, S^{lo}_{i}(t), \rho^{lo}_{i}(t)\right)}{P^{lo}_{i+1}(t) - P^{lo}_{i}(t)} \\
&= \frac{\Pi^{li}\left(t, S^{lo}_{i+1}(t), \rho^{lo}_{i}(t)\right) - \Pi^{li}\left(t, S^{lo}_{i}(t), \rho^{lo}_{i}(t)\right)}{P^{li}\left(t, S^{lo}_{i+1}(t)\right) - P^{li}\left(t, S^{lo}_{i}(t)\right)} \\
&\simeq \Delta^{li}_{t}
\end{aligned}$$

Analysis in a Markovian contagion model

- **Comparison of P&L increments** obtained using Δ^{lo} and Δ^{li}

$$\delta e^{li} = \delta e^{lo} + \left(\Delta^{li} - \Delta^{lo}\right) \delta P$$

- In a market where $\rho^{lo}_{i+1}(t) \geq \rho^{lo}_{i}(t)$ (**steep base correlation market**)
- $[0\% - b]$ equity tranche:

Market regime	**Rally**	**Sell-Off**
Slow	$(\delta e^{li})^{+} \leq \delta e^{lo}$	
Fast		$\delta e^{lo} \leq -(\delta e^{li})^{-}$

- $[a - 100\%]$ senior tranche:

Market regime	**Rally**	**Sell-Off**
Slow	$\delta e^{lo} \leq -(\delta e^{li})^{-}$	
Fast		$(\delta e^{li})^{+} \leq \delta e^{lo}$

Analysis in a Markovian contagion model

- **Comparison of P&L increments** obtained using Δ^{lo} and Δ^{li}

$$\delta e^{li} = \delta e^{lo} + \left(\Delta^{li} - \Delta^{lo}\right) \delta P$$

- In a market where $\rho^{lo}_{i+1}(t) \leq \rho^{lo}_i(t)$ (**flat base correlation market**)
- $[0\% - b]$ equity tranche:

Market regime	Rally	Sell-Off
Slow	$0 \leq \delta e^{lo} \leq \delta e^{li}$	
Fast		$\delta e^{li} \leq \delta e^{lo} \leq 0$

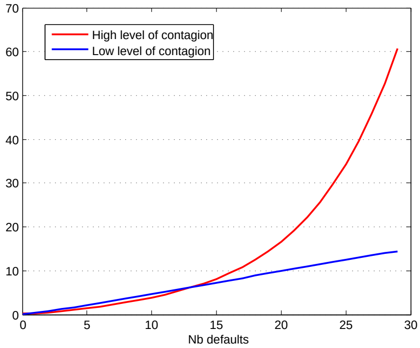
- $[a - 100\%]$ senior tranche:

Market regime	Rally	Sell-Off
Slow	$\delta e^{li} \leq \delta e^{lo} \leq 0$	
Fast		$0 \leq \delta e^{lo} \leq \delta e^{li}$

- Δ^{lo} provides a better hedge than Δ^{li}

Numerical results

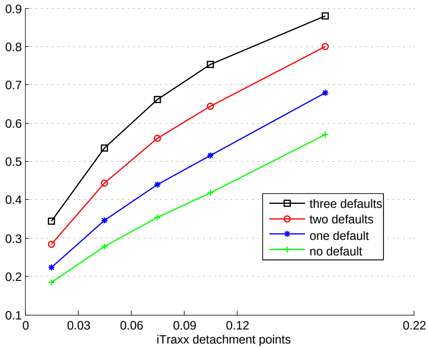
- Sell-protection position in a [0-3%] equity tranche
- We numerically compare hedging performance of Δ^{li} and Δ^{lo} in two different markets:
  - One with a high increase of contagion (red)
  - One with a low increase of contagion (blue)



Numerical results (high contagion market)

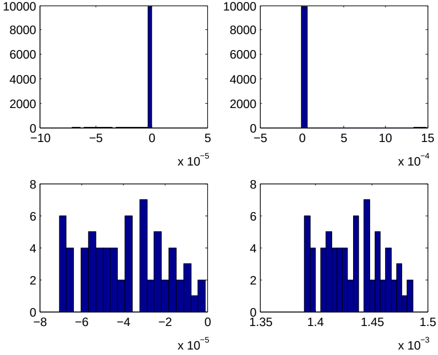
- The high contagion market features an increase of base correlation at the arrival of defaults:

$$\rho_{i+1}^{lo}(t) \geq \rho_i^{lo}(t)$$



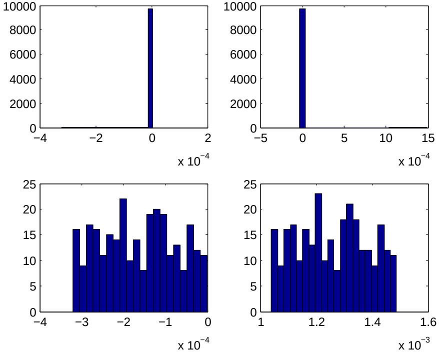
Numerical results (high contagion market)

- **Histogram of P&L increments** using Δ^{lo} (left) and Δ^{li} (right)
  - Hedging period: $[0, 0.02]$ (one week)
  - $\bar{N} = 10000$ trajectories



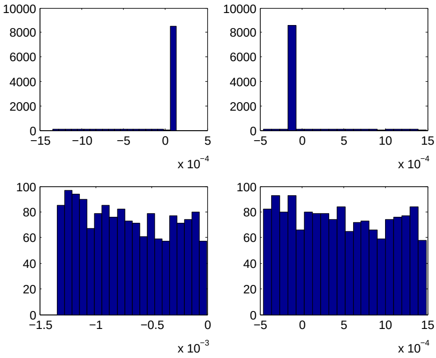
Numerical results (high contagion market)

- **Histogram of P&L increments** using Δ^{lo} (left) and Δ^{li} (right)
  - Hedging period: $[0, 0.09]$ (one month)
  - $\bar{N} = 10000$ trajectories



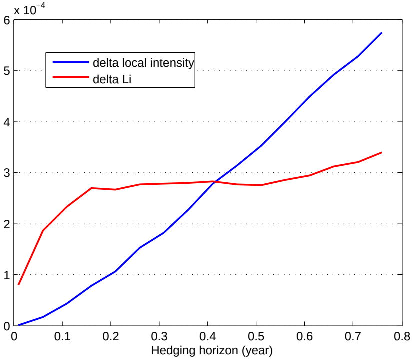
Numerical results (high contagion market)

- **Histogram of P&L increments** using Δ^{lo} (left) and Δ^{li} (right)
  - Hedging period: $[0, 0.5]$ (one semester)
  - $\bar{N} = 10000$ trajectories



Numerical results (high contagion market)

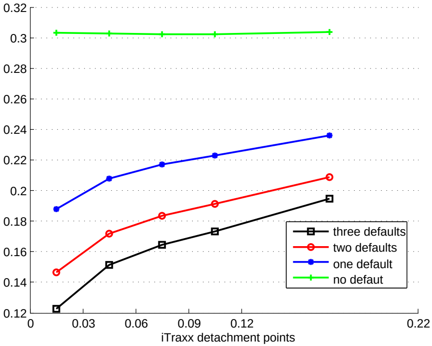
- Standard deviation of P&L increments function of the hedging horizon
  - $\bar{N} = 10000$ simulations at each time step
  - $\sigma(\delta P\&L)$ using Δ^{lo} in blue
  - $\sigma(\delta P\&L)$ using Δ^{li} in red



Numerical results (low contagion market)

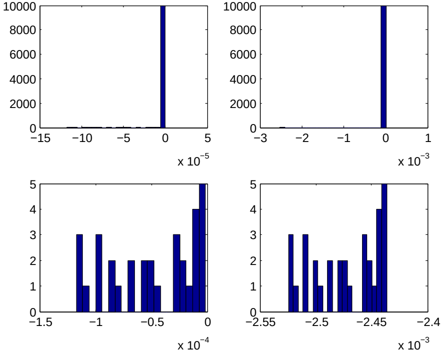
- The low contagion market features a decrease of base correlation at the arrival of defaults:

$$\rho_{i+1}^{lo}(t) \leq \rho_i^{lo}(t)$$



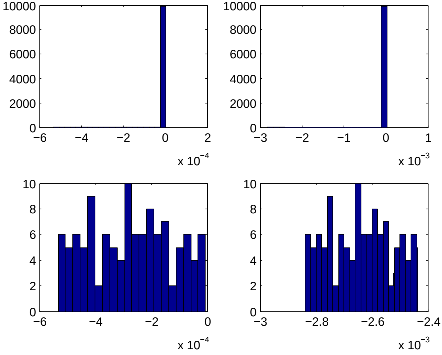
Numerical results (low contagion market)

- **Histogram of P&L increments** using Δ^{lo} (left) and Δ^{li} (right)
  - Hedging period: $[0, 0.02]$ (one week)
  - $\bar{N} = 10000$ trajectories



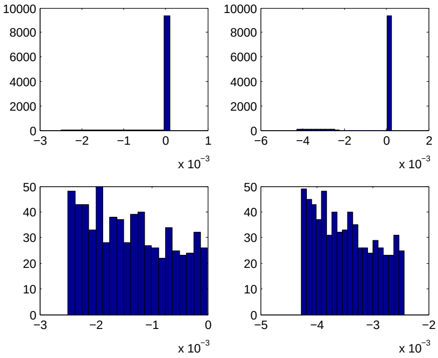
Numerical results (low contagion market)

- **Histogram of P&L increments** using Δ^{lo} (left) and Δ^{li} (right)
  - Hedging period: $[0, 0.09]$ (one month)
  - $\bar{N} = 10000$ trajectories



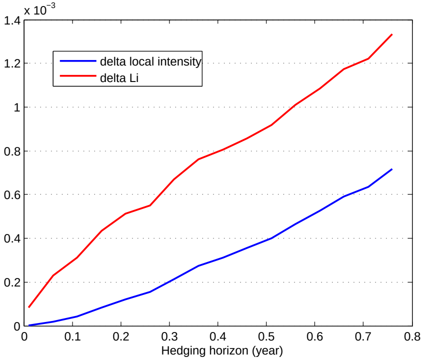
Numerical results (low contagion market)

- **Histogram of P&L increments** using Δ^{lo} (left) and Δ^{li} (right)
  - Hedging period: $[0, 0.5]$ (one semester)
  - $\bar{N} = 10000$ trajectories



Numerical results (low contagion market)

- Standard deviation of P&L increments function of the hedging horizon
  - $\bar{N} = 10000$ simulations at each time step
  - $\sigma(\delta P\&L)$ using Δ^{lo} in blue
  - $\sigma(\delta P\&L)$ using Δ^{li} in red



Hedging with individual CDS spreads

- Hedging with individual CDS may perform a better hedge (than hedging with the index)
  - heterogeneous portfolio where some individual spreads are suddenly widening
  - equity tranche very sensitive to idiosyncratic risk
- Obviously, hedging with single name sensitivities is beyond the reach of a pure top model
- **Future research**: Comparison of hedge performance with individual CDS contracts when hedging strategies are computed
  - using the market standard hedging method (spread-deltas in a base correlation approach): **bottom-up approach**
  - using a pure top model associated with a thinning procedure: **top-down approach**

Top-down approach

- In pure top model, the flow of information is only driven by the cumulative loss process ($\mathcal{H}_t = \sigma(L_s, s \leq t)$)
  - Given $\mathcal{H}_t$, we can only forecast the timing of defaults up to time t: ordered default times are $\mathcal{H}$-stopping times
  - But: lose of information related to the defaulters' identities
- Starting from a top model, Giesecke and Goldberg(2005) propose to recover single name information using a random thinning procedure
- The idea is to allocate a fraction of the loss intensity to each individual name with the constraint that the individual CDS spreads in the portfolio are matched

Set-up

- $\tau_1, \ldots, \tau_n$: default time, $N_t = \sum_{i=1}^n 1_{\{\tau_i \leq t\}}$
- Let us define $\mathcal{H} = \{\mathcal{H}_t\}$, where:

$$\mathcal{H}_t = \sigma(N_s, s \leq t)$$

- $\tau^{(1)} < \ldots < \tau^{(n)}$: ordered default time
- Let us define by $\mathcal{I} = \{\mathcal{I}_t\}$ the defaulter's identity filtration, where

$$\mathcal{I}_t := \sigma\left(I_{ij} \mid i = 1, \ldots, n;\ j = 1, \ldots, N_t\right)$$

- $\mathcal{F} = \{\mathcal{F}_t\}$: background filtration that contains the external market information.
- $\mathcal{G} = \{\mathcal{T}_t\}$: largest filtration

$$\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{I}_t \vee \mathcal{F}_t$$

Compensator of ordered default times

- $\tau^{(1)} < \ldots < \tau^{(n)}$ ordered default times are $\mathcal{G}$-stopping time
- $\Lambda^{(1)}, \ldots, \Lambda^{(n)}$: $\mathcal{G}$-compensators of $\tau^{(1)}, \ldots, \tau^{(n)}$
- Λ: $\mathcal{G}$-compensator of N
- Bielecki, Crépey, Jeanblanc(2008):

Proposition

For $t \geq 0$, $\Lambda_t^{(i)} = \Lambda_{t \wedge \tau^{(i)}} - \Lambda_{t \wedge \tau^{(i-1)}}$, $i = 1, \ldots, n$

- $\tau^{(1)} < \ldots < \tau^{(n)}$ are $\mathcal{H}$-stopping time

Random Thinning

- $\tau_1, \dots, \tau_n$ are $\mathcal{G}$-stopping times
- $\Lambda_1, \dots, \Lambda_n$: $\mathcal{G}$-compensators of $\tau_1, \dots, \tau_n$
- Λ: $\mathcal{G}$-compensator of N

$$\Lambda = \sum_{i=1}^{n} \Lambda_i$$

- Giesecke and Goldberg(2005):

Proposition

There exists $\mathcal{G}$-predictable non-negative processes Z_i, $i = 1, \dots, n$ (Z-factors) such that $\sum_{i=1}^{n} Z_i = 1$ and

$$\Lambda_i = \int_0 Z_{i,t} d\Lambda_t, \quad i = 1, \dots, n.$$

Random Thinning

- $\lambda_{i,t}$: $\mathcal{G}$-intensities of τ_i, $i = 1, \ldots, n$
- λ_t: $\mathcal{G}$-intensity of N
- $\lambda_{i,t} = Z_{i,t}\lambda_t$, $i = 1, \ldots, n$ and $\sum_{i=1}^n Z_i = 1$
- $Z_{i,t}$ is the conditional probability that name i is the next defaulter given an imminent default in the interval $[t, t + dt[$:

$$Z_{i,t} = \sum_{j=1}^{n} \mathbb{P}\left(\tau^{(j)} = \tau_i \mid t < \tau^{(j)} \leq t + dt, \, \mathcal{G}_t\right) 1_{\left\{\tau^{(j-1)} < t \leq \tau^{(j)}\right\}}$$

- **Top-Down matrix**: $P(t) = \left(p_{i,j}(t)\right)_{1 \leq i,j \leq n}$

$$p_{i,j}(t) = \mathbb{P}\left(\tau^{(j)} = \tau_i \mid t < \tau^{(j)} \leq t + dt, \, \mathcal{G}_t\right)$$

- **Consistency condition**: $\sum_{i=1}^n p_{i,j}(t) = 1$, $j = 1, \ldots, n$

Random draws without replacement

- Approach proposed by Halperin and Tomecek (2008)
- TD matrix piecewise constant in t, only change at default times
- $t = 0$, no default $p^0_{i,j} = \mathbb{P}\left(\tau^{(j)} = \tau_i\right)$ (inputs)
- Simulation of $\tau^{(1)}, \dots, \tau^{(n)}$ (or N) in the "small filtration", i.e $\mathcal{H}$
- At $t = \tau^{(1)}$ (first jump of N): independent simulation of the defaulter identity $I_1 \in \{1, 2, \dots, n\}$ according to the distribution:

$$\mathbb{P}(I_1 = i) = p^0_{i,1}, \; i = 1, \dots, n$$

- Update the TD matrix $p^0_{i,j} \to p^1_{i,j} = \mathbb{P}\left(\tau^{(j)} = \tau_i \mid \tau^{(1)}, I_1\right)$

$$p^1_{i,j} = \begin{cases} 0 & i = I_1, \; j = 1, \dots, n \\ 0 & j = 1, \; i = 1, \dots, n \\ \frac{p^0_{i,j}}{1 - p^0_{I,j}} & i \neq I, \; j = 2, \dots, n \end{cases}$$

- **Practical issue**: if $N_t = \sum_{i=1}^n 1_{\{\tau_i \leq t\}}$ is Markov with respect to the "small filtration" $\mathcal{H}$, it is no more the case in the "large filtration" $\mathcal{G}$

Random draws with replacement

- **Pure top model**: homogeneous Markovian contagion model (local intensity)
- At the j-th jump of N: independent simulation of the defaulter identity $I_j \in \{1, 2, \ldots, n\}$ according to the distribution:

$$p_{i,j}, \text{ where } \sum_{i=1}^{n} p_{i,j} = 1$$

- After the draw, I_j is replaced in the pool: TD matrix is not updated
- Denote by $B_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, n$ some independent Bernoulli random variables such that $\mathbb{E}[B_{i,j}] = p_{i,j}$
- We build n individual counting process $N_i(t)$ such that

$$N_{i,t} = \sum_{j=1}^{N_t} B_{i,j}, \ i = 1, \ldots, n$$

Random draws with replacement

- N_i cannot be identified with the "true" usual default process of i (single jump to default)
  - But here $\mathbb{E}[N_T \mid \mathcal{G}_t] = \mathbb{E}[N_T \mid \mathcal{H}_t] = \mathbb{E}[N_T \mid N_t]$
  - Tractable calibration to CDO tranches, individual CDS quotes
  - We hope that individual delta spreads are relevant ...