An extension of Davis and Lo’s contagion model

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Wednesday presentation by Sheri Markose illustrates the usefulness of network models to better understand systemic risk and default contagion amongst financial institutions.

Das et al. (2007) or Azizpour and Giesecke (2008): Conditional independence assumption with no contagion effect is rejected by historical default data.


We present a multi-period extension of Davis and Lo’s contagion model.
Literature

In the spirit of Davis and Lo's contagion model:

- First models: Davis and Lo (2001) and Jarrow and Yu (2001)

Other contagion models in the credit risk field:

- Copula: Schönbucher and Schubert (2001)
Modeling of credit contagion for a pool of defaultable entities

- One-period model
- Credit references may default either directly or as a consequence of a contagion effect

Example: Portfolio with 3 credit references
End of the period: direct default
End of the period: default by contagion (one possibility)
End of the period: default by contagion (another possibility)
One-period contagion model

- $n$: number of credit references
- $X_i$: direct default indicator of name $i$.
- $C_i$: indirect default indicator of name $i$.
- $D_i$: default indicator (direct or indirect) such that:
  \[ D_i = X_i + (1 - X_i)C_i \]

where:

\[ C_i = 1 - \prod_{j \neq i} (1 - X_j Y_{ji}) \quad \text{i.e.,} \]

\[ C_i = 1 \quad \text{at least one } x_j Y_{ji} = 1, \ j = 1, \ldots, n \]

- $Y_{ji}, \ i, j = 1, \ldots, n$ are Bernoulli random variables
- $Y_{ji} = 1$ if the contagion link is activated from name $j$ to name $i$. 
Davis and Lo’s contagion model

\[ N = \sum_{i=1}^{n} D_i : \text{total number of defaults} \]

Distribution of total number of defaults (Davis and Lo)

\[
\begin{align*}
P [N = k] &= C_n^k p^k (1 - p)^{n-k} (1 - q)^{k(n-k)} + \\
&\quad C_n^k \sum_{i=1}^{k-1} C_i^k p^i (1 - p)^{n-i} (1 - (1 - q)^i)^{k-i} (1 - q)^{i(n-k)}
\end{align*}
\]

Under the assumptions:

- \( X_i, i = 1, \ldots, n \) : iid Bernoulli with parameter \( p \)
- \( Y_{ij}, i, j = 1, \ldots, n \) : iid Bernoulli with parameter \( q \)
- One default alone may trigger a contamination effect
- A name that has been infected cannot contaminate other names (no chain-reaction effect)
Extension of Davis and Lo’s contagion model

**Dominos Effect**

Two defaults required to trigger a contagion effect
Multi-period contagion model: $t = 0, 1, 2, \ldots, T$, in period $[t, t+1]$:

- $n$: number of credit references
- $X^i_t$: direct default indicator of name $i$
- $C^i_t$: indirect default indicator of name $i$
- $D^i_t$: default indicator (direct or indirect) such that:

$$D^i_t = D^i_{t-1} + (1 - D^i_{t-1})[X^i_t + (1 - X^i_t)C^i_t]$$

where

$$C^i_t = f \left( \sum_{j \in F_t} Y^{ji}_t \right)$$

- $Y^{ji}_t$, $i, j = 1, \ldots, n$ are Bernoulli random variables such that $Y^{ji}_t = 1$ if name $j$ may infect name $i$ between $t$ and $t+1$
- $F_t$ is the set of names that are likely to infect other names between $t$ and $t+1$
- $f$ is a contamination trigger function, for example $f = \mathbb{1}_{x \geq 1}$ (Davis and Lo) or $f = \mathbb{1}_{x \geq 2}$
$N_t = \sum_{i=1}^{n} D_t^i$: total number of defaults at time $t$

Main result

$$P[N_t = r] = \sum_{k=0}^{r} P[N_{t-1} = k] C_{n-k}^{r-k} \sum_{\gamma=0}^{r-k} C_{r-k}^{\gamma} \cdot \sum_{\alpha=0}^{n-k-\gamma} C_n^{\alpha} \mu_{\gamma+\alpha, t} \sum_{j=0}^{n-r} C_n^{j-r} (-1)^{j+r-k-\gamma} \xi_{j+r-k-\gamma, t}$$

Under the assumptions:

- $X_t^i, i = 1, \ldots, n$ are conditionally independent Bernoulli random variables with the same marginal distribution and $X_t = (X_t^1, \ldots, X_t^n)$, $t = 1, \ldots, T$ are independent vectors.
- $Y_t^{i,j}, i, j = 1, \ldots, n$ are conditionally independent Bernoulli random variables with the same marginal distribution and $Y_t = (Y_t^{i,j})_{1 \leq i, j \leq n}$, $t = 1, \ldots, T$ are independent vectors.
- $(X_t)_{t=1,\ldots,T}$ and $(Y_t)_{t=1,\ldots,T}$ are independent.
Calibration on 5-years iTraxx tranche quotes

Cash-flows of CDO tranches driven by the aggregate loss process:

\[ L_t = \sum_{i=1}^{n} (1 - R_i) D_t^i \]

where \( R_i \) is the recovery rate associated with name \( i \).
We restrict ourselves to the case where for all $t$:

- $X^i_t \sim \text{Bernoulli}(\Theta)$ where $\Theta \sim \text{Beta}$, $\mathbb{E}[\Theta] = p$ and $\text{Var}(\Theta) = \sigma^2$, $i = 1, \ldots, n$
- $Y^{ij}_t$ are iid Bernoulli random variables with mean $q$, i.e., $Y^{ij}_t \sim \text{Bernoulli}(q)$, $i, j = 1, \ldots, n$
- Only one default is required to trigger a default by contagion

Moreover:

- $n = 125$, $r = 3\%$ (short-term interest rate)
- $R_i = R = 40\%$ for any $i = 1, \ldots, n$

$$L_t = (1 - R)N_t$$

- Computation of CDO tranche price only requires marginal loss distributions at several time horizons
Least square calibration procedure: Find $\alpha^* = (p^*, \sigma^*, q^*)$ which minimizes:

$$RMSE(\alpha) = \sqrt{\frac{1}{6} \sum_{i=1}^{6} \left( \frac{\tilde{s}_i - s_i(\alpha)}{\tilde{s}_i} \right)^2}.$$ 

where

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<tr>
<th>Index</th>
<th>0%-3%</th>
<th>3%-6%</th>
<th>6%-9%</th>
<th>9%-12%</th>
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<td>$s_3(\alpha)$</td>
<td>$s_4(\alpha)$</td>
<td>$s_5(\alpha)$</td>
<td>$s_0(\alpha)$</td>
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Four calibration procedures:

- **Calibration 1**: All available market spreads are included in the fitting
- **Calibration 2**: The equity [0%-3%] tranche spread is excluded
- **Calibration 3**: Both equity [0%-3%] tranche and CDS index spreads are excluded
- **Calibration 4**: All tranche spreads are excluded except equity tranche and CDS index spreads.

Two calibration dates before and during the credit crisis:

- 31 August 2005
- 31 March 2008
### Calibration on 5-years iTraxx tranche quotes

31 August 2005

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### Annual scaled optimal parameters

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Calibration on 5-years iTraxx tranche quotes

31 March 2008

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Annual scaled optimal parameters

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Conclusion

We propose a multi-period extension of Davis and Lo’s contagion model that accounts for

- possibly dominos or chain reaction effect
- flexible contagion mechanism (ex: more than one default required to trigger a contamination)

We provide a recursive formula for the distribution of the number of defaults at different time horizons

- When direct defaults and contagion events are conditionally independent

The multi-period setting is required to price synthetic CDO tranches

- The contagion parameter has a significant impact on the model ability to fit CDO tranche quotes