Dynamic Modeling of Portfolio Credit Risk with Common Shocks

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Introduction

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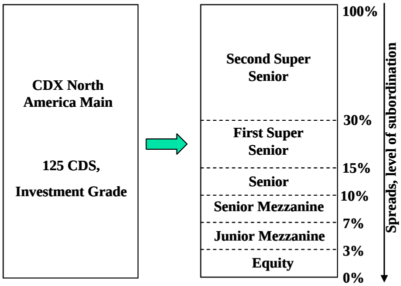
Main issue: hedging of portfolio credit derivatives



- Cash-flows driven by the realized path of the aggregate loss process

$$L_t = \frac{1}{n} \sum_{i=1}^{n} (1 - R_i) H_t^i$$

where R_i is the recovery rate and H_t^i is the default indicator of obligor i

Introduction

Hedging using the one-factor Gaussian copula model?

Advantages:

- **Bottom-up model**: account for dispersion of default risk among names in the portfolio
- **Copula construction of default times**: Calibration of CDS spreads and CDO tranche quotes can be made using two separate numerical procedures
- **Factor model**: fast algorithms to compute aggregate loss distribution

Drawbacks:

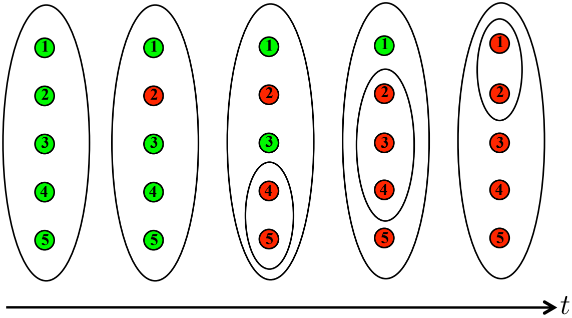
- **Static model**
- **Base correlation approach** unable to describe consistently the dependence structure of default times

Dynamic model of portfolio credit risk

Simultaneous default model

- Defaults are the consequence of triggering-events affecting simultaneously pre-specified groups of obligors

Example: $n = 5$ and $\mathcal{Y} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{4,5\}, \{2,3,4\}, \{1,2\}\}$.



Dynamic model of portfolio credit risk

- $\{1, \ldots, n\}$ set of credit references
- $\mathcal{Y} = \{\{1\}, \ldots, \{n\}, I_1, \ldots, I_m\}$ pre-specified groups of obligors
- $\lambda_Y = \lambda_Y(t)$ deterministic intensity function of the triggering-event associated with group $Y \in \mathcal{Y}$
- $\mathbf{H}_t = (H_t^1, \ldots, H_t^n)$ defined as multivariate continuous-time Markov chain in $\{0,1\}^n$ such that for $\mathbf{k}, \mathbf{m} \in \{0,1\}^n$:

$$\mathbb{P}\left(\mathbf{H}_{t+dt} = \mathbf{m} \mid \mathbf{H}_t = \mathbf{k}\right) = \sum_{Y \in \mathcal{Y}} \lambda_Y(t) \mathbf{1}_{\{\mathbf{k}^Y = \mathbf{m}\}} dt$$

  where $\mathbf{k}^Y$ is obtained from $\mathbf{k} = (k_1, \ldots, k_n)$ by replacing the components k_j, $j \in Y$, by number one. ex: $(0,1,0,0)^{\{1,2,4\}} = (1,1,0,1)$
- $\mathcal{F}_t = \sigma(\mathbf{H}_u, u \leq t)$ natural filtration of $\mathbf{H}$

Dynamic model of portfolio credit risk

Example: $n = 2$, $\mathcal{Y} = \{\{1\}, \{2\}, \{1,2\}\}$. $\mathbf{H}_t = (H_t^1, H_t^2)$ is a multivariate continuous-time Markov chain with space set $\{(0,0), (1,0), (0,1), (1,1)\}$ and generator matrix

$$\begin{array}{c c} & \begin{array}{cccc} (0,0) & (1,0) & (0,1) & (1,1) \end{array} \\ \begin{array}{c} (0,0) \\ (1,0) \\ (0,1) \\ (1,1) \end{array} & \begin{pmatrix} - & \lambda_{\{1\}} & \lambda_{\{2\}} & \lambda_{\{1,2\}} \\ 0 & - & 0 & \lambda_{\{2\}} + \lambda_{\{1,2\}} \\ 0 & 0 & - & \lambda_{\{1\}} + \lambda_{\{1,2\}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- Obligor 1 defaults with intensity $\lambda_{\{1\}} + \lambda_{\{1,2\}}$ regardless of the state of the pool
- Obligor 2 defaults with intensity $\lambda_{\{2\}} + \lambda_{\{1,2\}}$ regardless of the state of the pool
- No contagion effect : Past defaults do not have any effect on intensities of surviving names

Dynamic model of portfolio credit risk

General case: Obligor i defaults with intensity $\eta_i(t) = \sum_{Y \in \mathcal{Y}} \lambda_Y(t) \mathbf{1}_{\{i \in Y\}}$

$$\mathbb{P}(H^i_{t+dt} - H^i_t = 1 \mid \mathcal{F}_t) = \mathbb{P}(H^i_{t+dt} - H^i_t = 1 \mid H^i_t) = (1 - H^i_t)\eta_i(t)dt$$

- Each default indicator H^i, $i = 1, \ldots, n$ is a Markov process with respect to $\mathcal{F}$

Pricing and Hedging

Separate calibration procedure of CDS-s and CDO tranches

For any $i = 1, \ldots, n$, the price at time t of a CDS referencing name i (European-type payoff):

$$\mathbb{E}\left[\Phi(H_T^i) \mid \mathcal{F}_t\right] = \mathbb{E}\left[\Phi(H_T^i) \mid H_t^1, \ldots, H_t^n\right] = \mathbb{E}\left[\Phi(H_T^i) \mid H_t^i\right]$$

Hedging CDO tranches with single-name CDS

- Derive price dynamics of CDO tranche and single-name CDS-s
- Computation of min-variance hedging strategies in this incomplete market model
- But: price of portfolio loss derivatives solves a large system of Kolmogorov backward equations that is numerically intractable at least for large portfolios ($n > 20$)

Common-Shock Model Interpretation

- For any pre-specified group $Y \in \mathcal{Y} = \{\{1\}, \dots, \{n\}, I_1, \dots, I_m\}$, we define
$$\tau_Y = \inf\left\{t \geq 0 \mid \int_0^t \lambda_Y(s)ds > E_Y\right\}$$
where E_Y, $Y \in \mathcal{Y}$ are independent and exponentially distributed random variables with parameter 1.
- τ_Y is the arrival time of shock Y that yields default of non-defaulted names in group Y
- Default time of name $i = 1, \dots, n$ defined by:
$$\widehat{\tau}_i = \min_{\{Y \in \mathcal{Y};\, i \in Y\}} \tau_Y$$

Common-Shock Model Interpretation

For all $t_1, \dots, t_n \geq 0$, the following relation holds
$$\mathbb{P}\left(\widehat{\tau}_1 > t_1, \dots, \widehat{\tau}_n > t_n\right) = \mathbb{P}\left(\tau_1 > t_1, \dots, \tau_n > t_n\right)$$
where $\tau_i := \inf\left\{t \geq 0 \mid H_t^i = 1\right\}$ is the default time of name i in the Markovian model

Common-Shock Model Interpretation

Using fast recursion procedure for pricing and hedging CDO tranches

- Thanks to the common-shock model interpretation:

$$L_t = \frac{1}{n}\sum_{i=1}^{n}(1-R_i)\mathbf{1}_{\{\tau_i \le t\}} \overset{d}{=} \frac{1}{n}\sum_{i=1}^{n}(1-R_i)\mathbf{1}_{\{\widehat{\tau}_i \le t\}}$$

- where $\mathbf{1}_{\{\widehat{\tau}_1 \le t\}}, \ldots, \mathbf{1}_{\{\widehat{\tau}_n \le t\}}$ are conditionally independent Bernoulli's given common-shock indicators $\mathbf{1}_{\{\tau_{I_1} \le t\}}, \ldots, \mathbf{1}_{\{\tau_{I_m} \le t\}}$
- For any state of the Markov process $\mathbf{H}_t$, there exists an "equivalent" common-shock model that matches joint distribution of default times for non-defaulted names.
- Computation of min-variance hedging strategies is also tractable

Common-Shock Model Interpretation

Two-step calibration procedure

First step:

- CDS spread of name i at time $t = 0$ can be expressed as a function of survival probabilities $\mathbb{P}(\tau_i > t) = \exp\left(-\int_0^t \eta_i(u)du\right)$ where

$$\eta_i(u) = \lambda_{\{i\}}(u) + \sum_{k=1}^{m} \lambda_{I_k}(u)\mathbf{1}_{\{i \in I_k\}}$$

- Marginal default intensities η_i, $i = 1, \ldots, n$, can be calibrated on single-name CDS curves using a standard bootstrap procedure

Second step:

- Common-shock intensities λ_{I_k}, $k = 1, \ldots, m$ are calibrated on CDO tranche quotes using the recursion algorithm

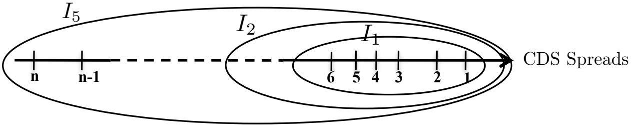
Calibration on CDX index

Data set: 5-year CDX North-America IG index on 20 December 2007

- Quoted spreads at different pillars of the $n = 125$ index constituents
- Quoted spreads of standard tranches [0,3], [3,7], [7,10], [10,15], [15,30]

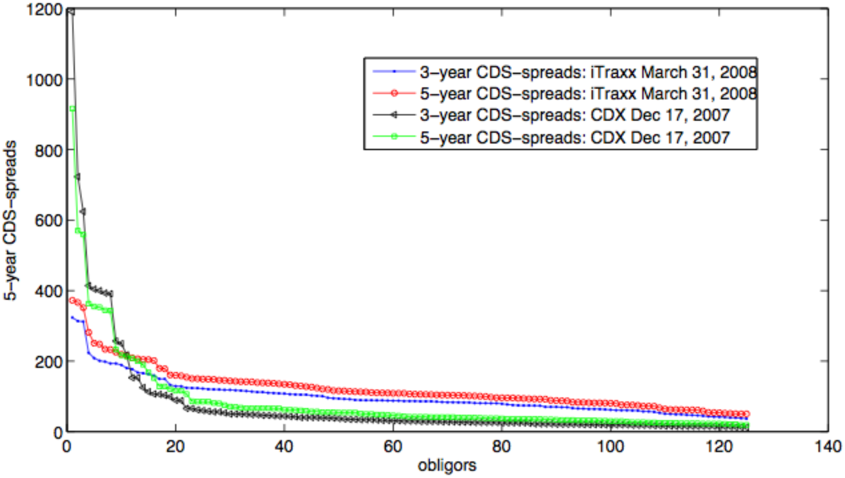
Model specification:

- Names are labelled with respect to decreasing level of spreads



- $m = 5$ groups $I_1 \subset \cdots \subset I_5$ such that $I_1 = \{1, \ldots, 6\}$, $I_2 = \{1, \ldots, 19\}$, $I_3 = \{1, \ldots, 25\}$, $I_4 = \{1, \ldots, 61\}$, $I_5 = \{1, \ldots, 125\}$
- Piecewise-constant intensities $\lambda_{\{1\}}, \ldots, \lambda_{\{125\}}$, $\lambda_{I_1}, \ldots, \lambda_{I_5}$ with grid points corresponding to CDS pillars
- Homogeneous and constant recovery rates: 40%
- Constant short-term interest rate: 3%

Calibration on CDX index



Calibration on CDX index

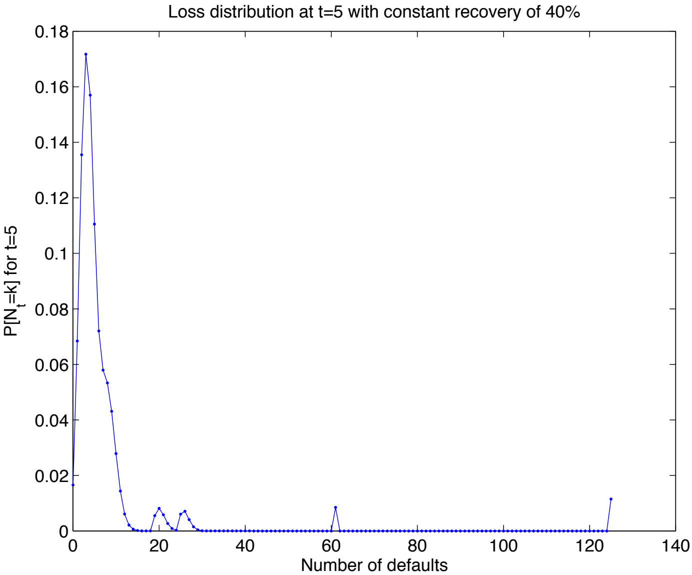
Calibration results:

Tranche	[0,3]	[3,7]	[7,10]	[10,15]	[15,30]
Model spread in bps	48.0701	254.0000	124.0000	61.0000	38.9390
Market spread in bps	48.0700	254.0000	124.0000	61.0000	41.0000
Abs. Err. in bps	0.0001	0.0000	0.0000	0.0000	2.0610
% Rel. Err.	0.0001	0.0000	0.0000	0.0000	5.0269

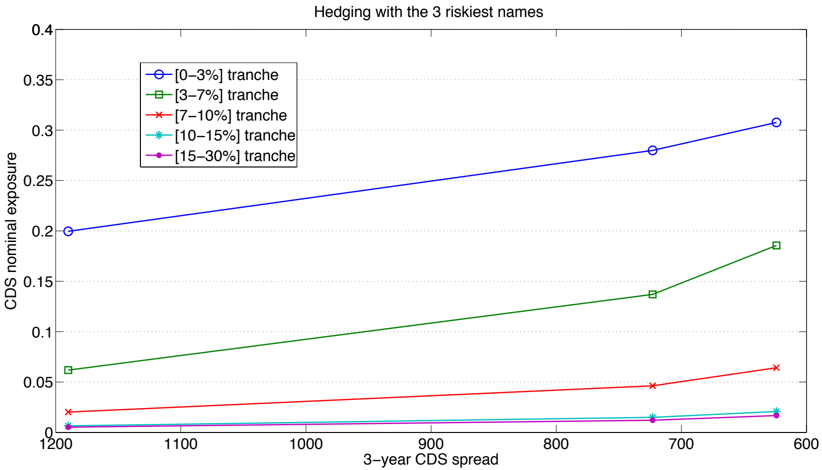
- Names in the set $I_5 \setminus I_4$ are excluded from the calibration constraints (they can only default within the Armageddon shock I_5)

Calibration on CDX index

Implied 5-year loss distribution:



Min-variance hedging strategies



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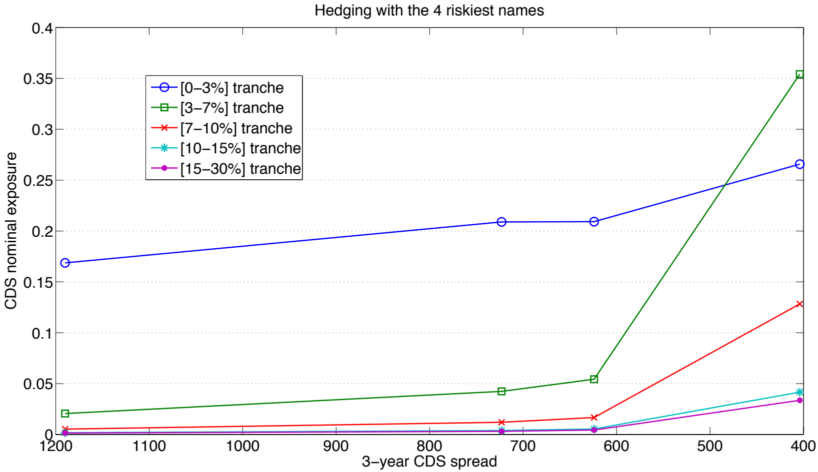

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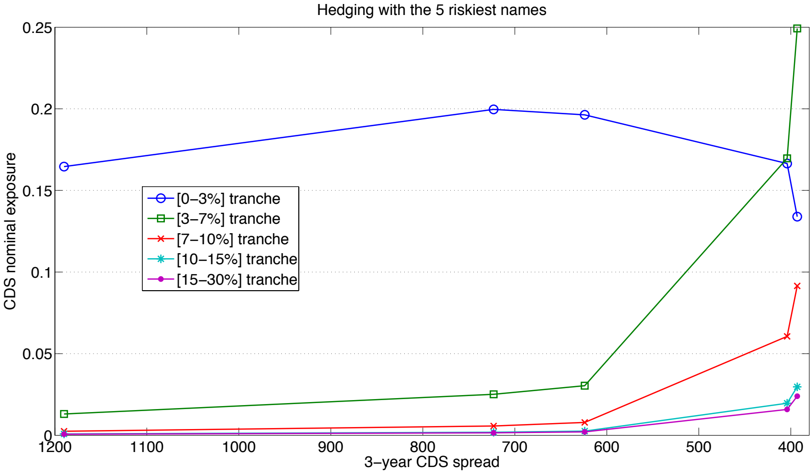
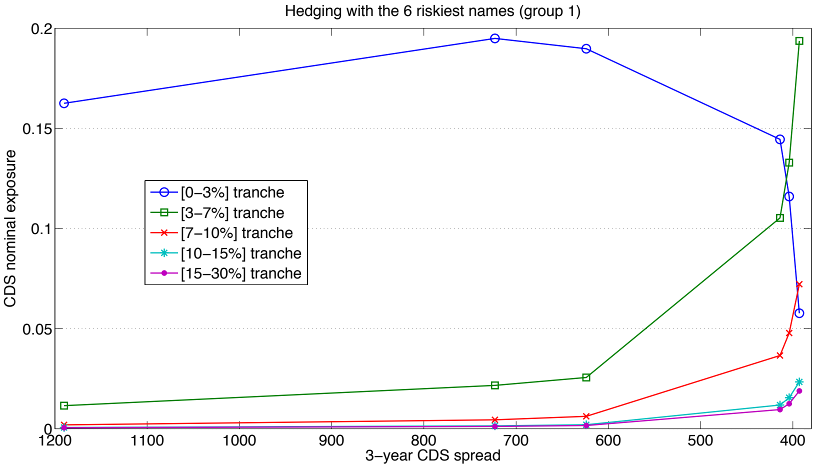


Min-variance hedging strategies



Thank you for your attention!

References

BIELECKI, T.R., VIDOZZI, A. AND VIDOZZI, L.: A Markov Copulae Approach to Pricing and Hedging of Credit Index Derivatives and Ratings Triggered Step–Up Bonds, *J. of Credit Risk*, 2008.

BRIGO, D., PALLAVICINI, A., TORRENTIAL, R.: Calibration of CDO Tranches with the Dynamical Generalized-Poisson Loss Model. *Working Paper*, 2006.

ELOUERKHAOUI, Y.: Pricing and Hedging in a Dynamic Credit Model. *International Journal of Theoretical and Applied Finance*, Vol. 10, Issue 4, 703–731, 2007.

LINDSKOG, F. AND MCNEIL, A. J.: Common Poisson Shock Models: Applications to Insurance and Credit Risk Modelling. *ASTIN Bulletin*, 33(2), 209-238, 2003