Pricing and Hedging Loss Derivatives in a Markovian Bottom-Up Model with Simultaneous Defaults

Areski Cousin
ISFA, Université Lyon 1

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Tom Bielecki, Areski Cousin, Stéphane Crépey and Alexander Herbertsson
Pricing and Hedging Portfolio Credit Derivatives in a Bottom-up Model with Simultaneous Defaults
Main issue: hedging of portfolio credit derivatives

- Cash-flows driven by the realized path of the aggregate loss process

\[ L_t = \frac{1}{n} \sum_{i=1}^{n} (1 - R_i) H_t^i \]

where \( R_i \) is the recovery rate and \( H_t^i \) is the default indicator of obligor \( i \)
Hedging using the one-factor Gaussian copula model?

**Advantages:**

- Bottom-up model: account for dispersion of default risk among names in the portfolio
- Copula construction of default times: Calibration of marginal default distributions and dependence parameters can be made using two separate numerical procedures
- Factor model: fast algorithms to compute marginal distributions of the loss process and hedging sensitivities

**Drawbacks:**

- Static model
- Base correlation approach unable to describe consistently the dependence structure of default times
In this paper, we construct a bottom-up Markovian model consisting of

- $\mathbf{X} = (X^1, \ldots, X^n)$ a vector of factor processes
- $\mathbf{H} = (H^1, \ldots, H^n)$ a vector of default indicator processes ($H^i_t = 1$ if default of name $i$ occurs before time $t$)
- $\mathcal{F}_t = \mathcal{F}^\mathbf{X,H}_t$

and with the following key features

- P1: $(\mathbf{X}, \mathbf{H})$ is a Markov process with respect to $\mathcal{F}$
- P2: Each pair $(X^i, H^i)$ is a Markov process with respect to $\mathcal{F}$
- P3: Obligors are likely to default simultaneously
- P4: Computation of both marginal loss distributions and dynamic hedging strategies can be achieved by fast numerical procedure
Simultaneous default model

- Defaults are the consequence of triggering-events affecting simultaneously pre-specified groups of obligors

Example: $n = 5$ and $\mathcal{Y} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{4, 5\}, \{2, 3, 4\}, \{1, 2\}\}$. 

\[ \begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 \\
& & & & & \\
\end{array} \]
Markovian model of portfolio credit risk

- $\{1, \ldots, n\}$ set of credit references
- $\mathcal{Y} = \{\{1\}, \ldots, \{n\}, I_1, \ldots, I_m\}$ pre-specified groups of obligors
- $\lambda_Y = \lambda_Y(t)$ deterministic intensity function of the triggering-event associated with group $Y \in \mathcal{Y}$
- $\mathbf{H}_t = (H^1_t, \ldots, H^n_t)$ defined as multivariate continuous-time Markov chain in $\{0, 1\}^n$ such that for $k, m \in \{0, 1\}^n$:

$$
P(\mathbf{H}_{t+dt} = m \mid \mathbf{H}_t = k) = \sum_{Y \in \mathcal{Y}} \lambda_Y(t) \mathbf{1}_{\{k^Y = m\}} dt
$$

where $k^Y$ is obtained from $k = (k_1, \ldots, k_n)$ by replacing the components $k_j, j \in Y$, by number one. ex: $(0, 1, 0, 0)^{\{1,2,4\}} = (1, 1, 0, 1)$
- $\mathcal{F}_t = \sigma(\mathbf{H}_u, u \leq t)$ natural filtration of $\mathbf{H}$
Example: $n = 2$, $\mathcal{Y} = \{\{1\}, \{2\}, \{1, 2\}\}$. $H_t = (H^1_t, H^2_t)$ is a multivariate continuous-time Markov chain with space set $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$ and generator matrix

$$
\begin{pmatrix}
(0, 0) & (1,0) & (0,1) & (1,1) \\
(0,0) & - & \lambda\{1\} & \lambda\{2\} & \lambda\{1,2\} \\
(1,0) & 0 & - & 0 & \lambda\{2\} + \lambda\{1,2\} \\
(0,1) & 0 & 0 & - & \lambda\{1\} + \lambda\{1,2\} \\
(1,1) & 0 & 0 & 0 & 0
\end{pmatrix}
$$

- Obligor 1 defaults with intensity $\lambda\{1\} + \lambda\{1,2\}$ regardless of the state of the pool
- Obligor 2 defaults with intensity $\lambda\{2\} + \lambda\{1,2\}$ regardless of the state of the pool
Markovian model of portfolio credit risk

General case: Obligor $i$ defaults with intensity $\eta_i(t) = \sum_{Y \in Y} \lambda_Y(t) 1_{\{i \in Y\}}$

$$\mathbb{P}(H^i_{t+dt} - H^i_t = 1 \mid \mathcal{F}_t) = \mathbb{P}(H^i_{t+dt} - H^i_t = 1 \mid H^i_t) = (1 - H^i_t) \eta_i(t) dt$$

- Each default indicator $H^i$, $i = 1, \ldots, n$ is a Markov process with respect to $\mathcal{F}$ (Property P2 is then satisfied)
- No contagion effect: Past defaults do not have any effect on intensities of surviving names
The latter construction can be extended to the case of stochastic intensity functions:

\[ \lambda_Y = \lambda_Y(t, X_t), \ Y \in \mathcal{Y} \]

where \( X_t = (X_t^1, \ldots, X_t^n) \) is a multivariate diffusion process:

\[
dX_t^i = b_i(t, X^i_t) \, dt + \sigma_i(t, X^i_t) \, dW^i_t, \ i = 1, \ldots, n
\]

- \( \mathbf{W} = (W^1_t, \ldots, W^n_t) \): \( n \)-dimensional Brownian motion with correlation matrix \( q(t) = (\rho_{i,j}(t))_{1 \leq i, j \leq n} \)

- \( b_i, \sigma_i \) are suitable drift and variance function-coefficients
Markov property of the model

Let $\mathcal{F} = \mathcal{F}^{X,H}$ be the natural filtration of $(X, H)$. The process $(X, H)$ is an $\mathcal{F}$-Markov process with generator $A$ given by

$$A_t u(t, x, k) = \sum_{1 \leq i \leq n} \left( b_i(t, x_i) \partial_{x_i} u(t, x, k) + \frac{1}{2} \sigma_i^2(t, x_i) \partial_{x_i}^2 u(t, x, k) \right)$$

$$+ \sum_{1 \leq i < j \leq n} \rho_{i,j}(t) \sigma_i(t, x_i) \sigma_j(t, x_j) \partial_{x_i, x_j} u(t, x, k)$$

$$+ \sum_{Y \in \mathcal{Y}} \lambda_Y(t, x) \left( u(t, x, k^Y) - u(t, x, k) \right)$$
The intensity of a jump of $H^i$ from $H^i_{t-} = 0$ to 1 is given by:

$$
\eta_i(t, X_t) = \lambda \{i\} (t, X_t) + \sum_{k=1}^{m} \lambda_{I_k}(t, X_t) \mathbf{1}_{\{i \in I_k\}}
$$

**Markov copula property**

Under the following conditions

- $\lambda \{i\}(t, x)$ only depends on $x = (x_1, \ldots, x_n)$ through $x_i, i = 1, \ldots, n$
- $\lambda_{I_k}(t, x)$ does not depend on $x, k = 1, \ldots, m$

for every $i = 1, \ldots, n$, the process $(X^i, H^i)$ is an $\mathcal{F}$-Markov process admitting the following generator

$$
A^i_t u_i(t, x_i, k_i) = b_i(t, x_i) \partial_{x_i} u_i(t, x_i, k_i) + \frac{1}{2} \sigma_i^2(t, x_i) \partial^2_{x_i} u_i(t, x_i, k_i)
+ \eta_i(t, x_i) (u_i(t, x_i, 1) - u_i(t, x_i, k_i))
$$

**Practical implication:** two-steps calibration procedure of single-name and multi-name products
Hedging CDO tranches using single-name CDS-s

Set of fundamental martingales (jump components)

- $H^Z_t$ is the indicator process of simultaneous default of names in the set $Z$, for every subset $Z$ of $\{1, \ldots, n\}$
- $Y_t = Y \cap \text{supp}^c(H_{t-})$ stands for the set of survivors of set $Y$ right before $t$, for every pre-specified group $Y \in \mathcal{Y}$

Set of fundamental martingales

The process $M^Z_t$ defined by

$$dM^Z_t := dH^Z_t - \ell_Z(t, X_t, H_{t-})dt$$

is a martingale with respect to $\mathcal{F}$, where the intensity function $\ell_Z(t, x, k)$ is such that

$$\ell_Z(t, X_t, H_{t-}) = \sum_{Y \in \mathcal{Y}; Y_t = Z} \lambda_Y(t, X_t)$$
Hedging CDO tranches using single-name CDS-s

### Itô formula

Given a “regular enough” function \( u = u(t, x, k) \), one has, for \( t \in [0, T] \),

\[
du(t, X_t, H_t) = \left( \partial_t + A_t \right) u(t, X_t, H_t) dt + \nabla u(t, X_t, H_t) \sigma(t, X_t) dW_t + \sum_{Z \in \mathcal{Z}_t} \delta u^Z(t, X_t, H_{t^-}) dM^Z_t
\]

where

- \( \sigma(t, x) \): diagonal matrix with diagonal \( (\sigma_i(t, x_i))_{1 \leq i \leq n} \)
- \( \nabla u(t, x, k) = (\partial_{x_1} u(t, x, k), \ldots, \partial_{x_n} u(t, x, k)) \)
- \( \delta u^Z(t, x, k) = u(t, x, k^Z) - u(t, x, k) \)
- \( \mathcal{Z}_t = \{Y_t; Y \in \mathcal{Y}\} \setminus \emptyset \): set of all non-empty sets of survivors of sets \( Y \) in \( \mathcal{Y} \) right before time \( t \)

**Martingale dimension:** \( n + 2^n \)
Price dynamics for single-name CDS-s (buy-protection position)

- $T$: contract maturity
- $S_i$: $T$-year contractual CDS-spread of obligor $i$
- $t_1 < \cdots < t_p = T$: premium payment dates, $h = t_j - t_{j-1}$ length between two premium payment dates (typically a quarter)
- $R_i$: recovery rate of obligor $i$

In the rest of the presentation, we will assume zero interest rates except for numerical results.
The price $P_t^i$ and the cumulative value $\hat{P}_t^i$ at time $t \in [0, T]$ of a single-name CDS on obligor $i$ are given by

$$P_t^i = 1_{\{\tau_i > t\}} v_i(t, X_t^i)$$

$$d\hat{P}_t^i = 1_{\{\tau_i > t\}} \partial_{x_i} v_i(t, X_t^i) \sigma_i(t, X_t^i) dW_t^i$$

$$+ \sum_{Z \in \mathcal{Z}_t} 1_{\{i \in Z\}} \left( 1 - R_i - v_i(t, X_t^i) \right) dM_t^Z$$

for a pre-default pricing function $v_i(t, x_i)$ such that

$$1_{\{\tau_i > t\}} v_i(t, X_t^i) = \mathbb{E}[(1 - R_i) 1_{\{t < \tau_i \leq T\}} - S_i h \sum_{t < t_j \leq T} 1_{\{\tau_i > t_j\}} |\mathcal{F}_t}]$$
Hedging CDO tranches using single-name CDS-s

**Price dynamics for CDO tranche** $[a, b]$ (buy-protection position)

- $T$: contract maturity
- $a$: attachement point, $b$: detachement point, $0 \leq a < b \leq 1$
- $S^{a,b}$: $T$-year contractual spread of CDO tranche $[a, b]$
- $t_1 < \cdots < t_p = T$: premium payment dates, $h = t_j - t_{j-1}$ length between two premium payment dates (typically a quarter)
- CDO tranche cash-flows are driven by the tranche loss process

\[
L_t^{a,b} = L_{a,b}(H_t) = (L_t - a)^+ - (L_t - b)^+
\]

where

\[
L_t = L_t(H_t) = \frac{1}{n} \sum_{i=1}^{n} (1 - R_i) 1_{\{\tau_i \leq t\}}
\]

is the credit loss process for the underlying portfolio
The price $\Pi_t$ and the cumulative value $\hat{\Pi}$ at time $t \in [0, T]$ of a CDO-tranche $[a, b]$ are given by

$$\Pi_t = u(t, X_t, H_t)$$

$$d\hat{\Pi}_t = \nabla u(t, X_t, H_t) \sigma(t, X_t) dW_t$$

$$+ \sum_{Z \in \mathcal{Z}_t} \left( L_{a,b}(H^Z_{t-}) - L_{a,b}(H_{t-}) + \delta u^Z(t, X_t, H_{t-}) \right) dM^Z_t$$

for a pricing function $u(t, x, k)$ such that

$$u(t, X_t, H_t) = \mathbb{E} \left[ L_{T}^{a,b} - L_{t}^{a,b} - S^{a,b} h \sum_{t < t_j \leq T} \left( b - a - L_{t_j}^{a,b} \right) \mid \mathcal{F}_t \right]$$

The pricing function $u(t, x, k)$ solves a very large system of Kolmogorov pde
Hedging portfolio: first $d$ single-name CDS-s and the savings account

The vector of cumulative values $\hat{P} = (\hat{P}^1, \ldots, \hat{P}^d)^T$ associated with the first $d$ CDS-s has the following dynamics:

$$d\hat{P}_t = \nabla v(t, X_t, H_t) \sigma(t, X_t) dW_t + \sum_{Z \in Z_t} \Delta v^Z(t, X_t, H_{t-}) dM^Z_t$$

where

- $\nabla v$ is a $d \times n$-matrix such that $\nabla v(t, x, k)^j_i = 1_{\{k_j = 0\}} \partial_{x_j} v_i(t, x_i)$, for every $1 \leq i \leq d$ and $1 \leq j \leq n$

- $\Delta v^Z(t, x, k)$ is a $d$-dimensional column vector equal to $(1_{\{1 \in Z, k_1 = 0\}} ((1 - R_1) - v_1(t, x_1)), \ldots, 1_{\{d \in Z, k_d = 0\}} ((1 - R_d) - v_d(t, x_d)))^T$
Hedging CDO tranches using single-name CDS-s

**Tracking error:** Process \((e_t)\) such that \(e_0 = 0\) and for \(t \in [0, T]\):

\[
d e_t = d \hat{\Pi}_t - \zeta_t d \hat{P}_t \\
= \left( \nabla u(t, X_t, H_t) - \zeta_t \nabla v(t, X_t, H_t) \right) \sigma(t, X_t) d W_t \\
+ \sum_{Z \in \mathcal{Z}_t} \left( \Delta u^Z(t, X_t, H_{t-}) - \zeta_t \Delta v^Z(t, X_t, H_{t-}) \right) d M^Z_t
\]

where

- \(\zeta_t = (\zeta^1_t, \ldots, \zeta^d_t)\) gives the positions held at time \(t\) in CDS 1, \ldots, \(d\)

- \(\nabla u(t, x, k) = (\partial x_1 u(t, x, k), \ldots, \partial x_n u(t, x, k))\)

- \(\Delta u^Z(t, x, k) = \delta^Z u(t, x, k) + L_{a,b}(k^Z) - L_{a,b}(k)\)
Hedging CDO tranches using single-name CDS-s

Min-variance hedging strategies

The min-variance hedging strategy $\zeta$ for the CDO-tranche $[a, b]$ is

$$\zeta_t = \frac{d\langle \hat{\Pi}, \hat{P} \rangle_t}{dt} \left( \frac{d\langle \hat{P} \rangle_t}{dt} \right)^{-1} = \zeta(t, X_t, H_t)$$

where $\zeta = (u, v)(v, v)^{-1}$, with

$$(u, v) = (\nabla u)\sigma^2(\nabla v)^T + \sum_{Y \in \mathcal{Y}} \lambda_Y \Delta u^Y (\Delta v^Y)^T$$

$$(v, v) = (\nabla v)\sigma^2(\nabla v)^T + \sum_{Y \in \mathcal{Y}} \lambda_Y \Delta v^Y (\Delta v^Y)^T$$
Example: $n = 5$ and $\mathcal{Y} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{4, 5\}, \{2, 3, 4\}, \{1, 2\}\}$. 

General case: In the common-shock model, individual default indicators are such that

$$\hat{H}_t^i := \max \left\{ \hat{H}_t^Y, Y \in \mathcal{Y}, i \in Y \right\}$$

where $\hat{H}_t^Y, Y \in \mathcal{Y}$ are independent $\{0, 1\}$–point processes with intensity $\lambda_Y$. 

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Pricing and Hedging Loss Derivatives in a Markovian Bottom-Up Model
In the rest of the presentation, we consider no spread risk $\mathcal{F}_t = \mathcal{F}_t^H$

**Main result**

- $\hat{\tau}_i := \inf \left\{ t \geq 0 \mid \hat{H}^i_t = 1 \right\}$, $i = 1, \ldots, n$: default times in the common-shock model
- $\tau_i := \inf \left\{ t \geq 0 \mid H^i_t = 1 \right\}$, $i = 1, \ldots, n$: default times in the Markovian model

**Proposition**

For all $t_1, \ldots, t_n \geq 0$, the following relation holds

$$\mathbb{P}(\hat{\tau}_1 > t_1, \ldots, \hat{\tau}_n > t_n) = \mathbb{P}(\tau_1 > t_1, \ldots, \tau_n > t_n)$$
Main result ($\mathcal{F}_t$–conditional version)

- $\text{supp}(H_t)$: set of all defaulted names at time $t$
- $\mathcal{V}_t = \{Y \in \mathcal{V} : Y \notin \text{supp}(H_t)\}$: set of pre-specified groups that contain at least one alive obligor
- $\hat{H}_t^i := \max \left\{ \hat{H}_t^Y, Y \in \mathcal{V}_t, i \in Y \right\}$: individual default processes in the $\mathcal{F}_t$–related common-shock model
- $\hat{\tau}_i(t) := \inf \left\{ \theta \geq t \mid \hat{H}_\theta^i = 1 \right\}$, $i \in \text{supp}^c(H_t)$: default times of surviving names in the $\mathcal{F}_t$–related common-shock model
- $\tau_i := \inf \left\{ \theta \geq t \mid H_\theta^i = 1 \right\}$, $i \in \text{supp}^c(H_t)$: default times of surviving names in the $\mathcal{F}_t$–conditional Markovian model

Proposition

Let $Z$ be a subset of $\{1, \ldots, n\}$. For every $t_1, \ldots, t_n \geq t$, one has on the set $\{Z = \text{supp}^c(H_t)\}$

$$\mathbb{P}(\hat{\tau}_i(t) > t_i, i \in Z) = \mathbb{P}(\tau_i > t_i, i \in \text{supp}^c(H_t) \mid \mathcal{F}_t)$$
Calibration of marginal default intensities on single-name CDS-s

- Individual shocks + Common shocks: \( \mathcal{Y} = \{\{1\}, \ldots, \{n\}, I_1, \ldots, I_m\} \)
- Price at time \( t = 0 \) of CDS \( i \) can be expressed as a function of \( \mathbb{E}[H^i_t] \), \( t = 0, \ldots, T \)

\[
\mathbb{E}[H^i_t] = \mathbb{P}(\tau_i > t) = 1 - \exp \left( -\int_0^t \eta_i(u)du \right)
\]

where

\[
\eta_i(u) = \lambda_{\{i\}}(u) + \sum_{k=1}^m \lambda_{I_k}(u) 1_{\{i \in I_k\}}
\]

- Marginal default intensities \( \eta_i, i = 1, \ldots, n \), can be calibrated on single-name CDS curves using a bootstrap procedure
Calibration of common-shocks intensities on CDO tranches

Pricing of CDO tranches only involves marginal loss distributions

Thanks to the common-shock model interpretation:

\[ L_t = \frac{1}{n} \sum_{i=1}^{n} (1 - R_i) H_t^i = \frac{1}{n} \sum_{i=1}^{n} (1 - R_i) \hat{H}_t^i \]

Conditionally on \((\hat{H}^I_1, \ldots, \hat{H}^I_m), \hat{H}_t^1, \ldots, \hat{H}_t^n\) are independent Bernoulli’s with random parameters

\[ p_t^i = \begin{cases} 1 & i \in \bigcup_{k=1}^m \{I_k ; \hat{H}^I_k = 1\} \\ 1 - \exp \left( - \int_0^t \lambda_{\{i\}}(u) du \right) & \text{else} \end{cases} \]

where

\[ \lambda_{\{i\}}(u) = \eta_i(u) - \sum_{k=1}^m \lambda_{I_k}(u) 1_{\{i \in I_k\}} \geq 0 \]
Fast convolution-recursion procedure for computing loss distribution

Let $N_t^{(k)} = \sum_{i=1}^{k} \hat{H}_t^i$, $k = 1, \ldots, n$

Let $q_t^{(k)}(i) = \mathbb{P}\left(N_t^{(k)} = i \mid \hat{H}_t^{I_1}, \ldots, \hat{H}_t^{I_m}\right)$, $i = 0, \ldots, k$

The following recursion procedure can be used to compute the conditional loss distribution starting from $k = 0$ and $q_t^{(0)}(0) = 1$

$$
\begin{align*}
q_t^{(k+1)}(0) &= (1 - p_t^{k+1}) \cdot q_t^{(k)}(0) \\
q_t^{(k+1)}(i) &= p_t^{k+1} \cdot q_t^{(k)}(i - 1) + (1 - p_t^{k+1}) \cdot q_t^{(k)}(i), \quad i = 1, \ldots, k \\
q_t^{(k+1)}(k + 1) &= p_t^{k+1} \cdot q_t^{(k)}(k)
\end{align*}
$$

This gives the time-$t$ conditional distribution $q_t^{(n)}$ of the total number of defaults $N_t := N_t^{(n)} = \sum_{i=1}^{n} \hat{H}_t^i$
Fast convolution-recursion procedure for computing loss distribution

- The computation of the unconditional loss distribution involves a summation over $2^m$ terms, i.e., as many terms as possible states of the vector $(\hat{H}_{t}^{I_1}, \ldots, \hat{H}_{t}^{I_m})$.
- In the case where the groups are nested, i.e., $I_1 \subset \cdots \subset I_m$, the collection of events $(A_t^k)_{k=0,\ldots,m}$ defined by

$$
\begin{align*}
A_t^0 &= \{ \hat{H}_{t}^{I_1} = 0, \ldots, \hat{H}_{t}^{I_m} = 0 \} \\
A_t^k &= \{ \hat{H}_{t}^{I_k} = 1, \hat{H}_{t}^{I_{k+1}} = 0, \ldots, \hat{H}_{t}^{I_m} = 0 \}, \quad k = 1, \ldots, m-1 \\
A_t^m &= \{ \hat{H}_{t}^{I_m} = 1 \}
\end{align*}
$$

forms a partition of $\Omega$. 

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Fast convolution-recursion procedure for computing loss distribution

- Since $A^k_t, k = 0, \ldots, m$ are disjoint events:
  
  $$
P(N_t = i) = \sum_{k=1}^{m} P(N_t = i \mid A^k_t) P(A^k_t), \quad i = 0, \ldots, n$$

- $P(N_t = i \mid A^k_t)$ can be computed thanks to the previous recursion procedure using the fact that $\hat{H}_t^1, \ldots, \hat{H}_t^n$ are conditionally independent Bernoulli’s given $A^k_t$, for every $k = 0, \ldots, m$

- As $\hat{H}_t^{I_1}, \ldots, \hat{H}_t^{I_m}$ are independent rv, the probability of the event $A^k_t$ satisfies
  
  $$
P(A^k_t) = \left(1 - \exp\left(-\int_0^t \lambda_{I_k}(u)du\right)\right) \prod_{j=k+1}^{m} \exp\left(-\int_0^t \lambda_{I_j}(u)du\right)$$
Data set: 5-year CDX North-America IG index on 20 December 2007
- Quoted spreads at different pillars of the $n = 125$ index constituents
- Quoted spreads of standard tranches [0,3], [3,7], [7,10], [10,15], [15,30]

Model specification:
- Names are labelled with respect to decreasing level of spreads

$m = 5$ groups $I_1 \subset \cdots \subset I_5$ such that $I_1 = \{1, \ldots, 6\}$, $I_2 = \{1, \ldots, 19\}$, $I_3 = \{1, \ldots, 25\}$, $I_4 = \{1, \ldots, 61\}$, $I_5 = \{1, \ldots, 125\}$

Piecewise-constant intensities $\lambda_{\{1\}}, \ldots, \lambda_{\{125\}}, \lambda_{I_1}, \ldots, \lambda_{I_5}$ with grid points corresponding to CDS pillars
- Homogeneous and constant recovery rates: 40%
- Constant short-term interest rate: 3%
Calibration on CDX index

![Graph showing 5-year CDS-spreads for different indices and obligors]

- **3-year CDS-spreads**: iTraxx March 31, 2008
- **5-year CDS-spreads**: iTraxx March 31, 2008
- **3-year CDS-spreads**: CDX Dec 17, 2007
- **5-year CDS-spreads**: CDX Dec 17, 2007
Calibration on CDX index

Calibration results:

<table>
<thead>
<tr>
<th>Tranche</th>
<th>[0,3]</th>
<th>[3,7]</th>
<th>[7,10]</th>
<th>[10,15]</th>
<th>[15,30]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model spread in bps</td>
<td>48.0701</td>
<td>254.0000</td>
<td>124.0000</td>
<td>61.0000</td>
<td>38.9390</td>
</tr>
<tr>
<td>Market spread in bps</td>
<td>48.0700</td>
<td>254.0000</td>
<td>124.0000</td>
<td>61.0000</td>
<td>41.0000</td>
</tr>
<tr>
<td>Abs. Err. in bps</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>2.0610</td>
</tr>
<tr>
<td>% Rel. Err.</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>5.0269</td>
</tr>
</tbody>
</table>

- Names in the set $I_5 \setminus I_4$ are excluded from the calibration constraints (they can only default within the Armageddon shock $I_5$)
Calibration on CDX index

Implied 5-year loss distribution:

Loss distribution at $t=5$ with constant recovery of 40%

$P[N_t=k]$ for $t=5$

Number of defaults

0 20 40 60 80 100 120 140

0 0.02 0.04 0.06 0.08 0.1 0.12 0.14

Figure 1: Portfolio loss distribution and log-distribution calibrated to all 5-year CDX tranche quotes and to the $k=6$ riskiest CDS curves on 17 December 2007.

4. Hedging

Adis tinguishing feature of the presented hedging loss derivatives by single-name instruments can be both theoretically sound (thanks to the bottom-up Markovian background) and tractable (thanks to the Marshall-Olkin copula interpretation), as we now illustrate.

4.1 Price Dynamics

In the present nil-rates environment, the (ex-dividend) price process of an asset simply refers to the risk-neutral conditional expectation of future asset cashflows, where the cumulative value (or gain) process boils down to the sum of the price process and of the cumulative cash-flows process. It is readily checked that the gain process is a martingale, as opposed to the price process.

The notation $Z_t$ is that of our Itô formula (5).

Proposition 4.1 (i) The price $P_i$ and the cumulative value $\hat{P}_i$ of the credit derivative on name $i$ are such that, for $t \in [0, T]$, $P_i(t) = (1 - H_i(t)) v_i(t, X_i(t))$

$\hat{P}_i(t) = (1 - H_i(t)) \frac{\partial}{\partial x_i} v_i(t, X_i(t)) \sigma_i(t, X_i(t)) dW_i(t) + \sum_{Z \in Z_t} \phi_{i}(t, X_i(t)) dM_Z(t)$

for a pre-default pricing function $v_i(t, x_i)$ such that $(1 - H_i(t)) v_i(t, X_i(t)) + H_i(t) \phi_i(t) = \mathbb{E}[\phi_i(\sum_{t < T \leq T} (1 - H_i(T)) + \phi_i(H_i(T)) | F_t]$;

(ii) The price process $\Pi$ and the cumulative value $\hat{\Pi}$ of the portfolio loss derivative are such that, for $t \in [0, T]$, $\Pi(t) = u(t, X_t, H_t)$

$\hat{\Pi}(t) = \nabla u(t, X_t, H_t) \sigma(t, X_t) dW_t + \sum_{Z \in Z_t} \delta u_Z(t, X_t, H_t-1) dM_Z(t)$

As a Doob martingale, namely, the conditional expectation process of a fixed random variable.
Min-variance hedging strategies

Hedging with the 3 riskiest names

- [0–3%] tranche
- [3–7%] tranche
- [7–10%] tranche
- [10–15%] tranche
- [15–30%] tranche

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Hedging with the 4 riskiest names

- [0–3%] tranche
- [3–7%] tranche
- [7–10%] tranche
- [10–15%] tranche
- [15–30%] tranche

3-year CDS spread vs. CDS nominal exposure

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Hedging with the 5 riskiest names

CDS nominal exposure

3-year CDS spread

- [0–3%] tranche
- [3–7%] tranche
- [7–10%] tranche
- [10–15%] tranche
- [15–30%] tranche
Hedging with the 6 riskiest names (group 1)

CDS nominal exposure

[0–3%] tranche
[3–7%] tranche
[7–10%] tranche
[10–15%] tranche
[15–30%] tranche

3-year CDS spread
Thank you for your attention!

