Pricing and Hedging Loss Derivatives in a Markovian Bottom-Up Model with Simultaneous Defaults

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Introduction

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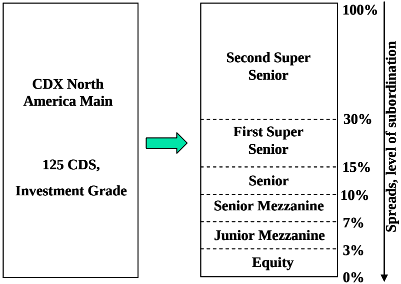
Main issue: hedging of portfolio credit derivatives



- Cash-flows driven by the realized path of the aggregate loss process

$$L_t = \frac{1}{n}\sum_{i=1}^{n}(1 - R_i)H_t^i$$

where R_i is the recovery rate and H_t^i is the default indicator of obligor i

Hedging using the one-factor Gaussian copula model?

Advantages:

- Bottom-up model: account for dispersion of default risk among names in the portfolio
- Copula construction of default times: Calibration of marginal default distributions and dependence parameters can be made using two separate numerical procedures
- Factor model: fast algorithms to compute marginal distributions of the loss process and hedging sensitivities

Drawbacks:

- Static model
- Base correlation approach unable to describe consistently the dependence structure of default times

Introduction

In this paper, we construct a bottom-up Markovian model consisting of

- $\mathbf{X} = (X^1, \ldots, X^n)$ a vector of factor processes
- $\mathbf{H} = (H^1, \ldots, H^n)$ a vector of default indicator processes ($H^i_t = 1$ iif default of name i occurs before time t)
- $\mathcal{F}_t = \mathcal{F}_t^{\mathbf{X},\mathbf{H}}$

and with the following key features

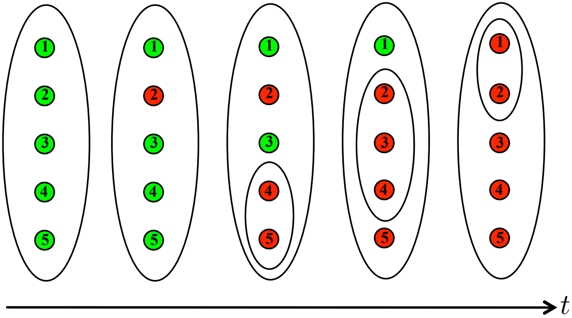
- P1: $(\mathbf{X}, \mathbf{H})$ is a Markov process with respect to $\mathcal{F}$
- P2: Each pair (X^i, H^i) is a Markov process with respect to $\mathcal{F}$
- P3: Obligors are likely to default simultaneously
- P4: Computation of both marginal loss distributions and dynamic hedging strategies can be achieved by fast numerical procedure

Markovian model of portfolio credit risk

Simultaneous default model

- Defaults are the consequence of triggering-events affecting simultaneously pre-specified groups of obligors

Example: $n = 5$ and $\mathcal{Y} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{4,5\}, \{2,3,4\}, \{1,2\}\}$.



Markovian model of portfolio credit risk

- $\{1,\ldots,n\}$ set of credit references
- $\mathcal{Y}=\{\{1\},\ldots,\{n\},I_1,\ldots,I_m\}$ pre-specified groups of obligors
- $\lambda_Y=\lambda_Y(t)$ deterministic intensity function of the triggering-event associated with group $Y\in\mathcal{Y}$
- $\mathbf{H}_t=(H_t^1,\ldots,H_t^n)$ defined as multivariate continuous-time Markov chain in $\{0,1\}^n$ such that for $\mathbf{k},\mathbf{m}\in\{0,1\}^n$:

$$\mathbb{P}\left(\mathbf{H}_{t+dt}=\mathbf{m}\mid\mathbf{H}_t=\mathbf{k}\right)=\sum_{Y\in\mathcal{Y}}\lambda_Y(t)\mathbf{1}_{\{\mathbf{k}^Y=\mathbf{m}\}}dt$$

  where $\mathbf{k}^Y$ is obtained from $\mathbf{k}=(k_1,\ldots,k_n)$ by replacing the components k_j, $j\in Y$, by number one. ex: $(0,1,0,0)^{\{1,2,4\}}=(1,1,0,1)$
- $\mathcal{F}_t=\sigma(\mathbf{H}_u\,,u\leq t)$ natural filtration of $\mathbf{H}$

Markovian model of portfolio credit risk

Example: $n = 2$, $\mathcal{Y} = \{\{1\},\{2\},\{1,2\}\}$. $\mathbf{H}_t = (H_t^1, H_t^2)$ is a multivariate continuous-time Markov chain with space set $\{(0,0),(1,0),(0,1),(1,1)\}$ and generator matrix

$$\begin{array}{c|c} & \begin{array}{cccc} (0,0) & (1,0) & (0,1) & (1,1) \end{array} \\ \begin{array}{c} (0,0) \\ (1,0) \\ (0,1) \\ (1,1) \end{array} & \begin{pmatrix} - & \lambda_{\{1\}} & \lambda_{\{2\}} & \lambda_{\{1,2\}} \\ 0 & - & 0 & \lambda_{\{2\}} + \lambda_{\{1,2\}} \\ 0 & 0 & - & \lambda_{\{1\}} + \lambda_{\{1,2\}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- Obligor 1 defaults with intensity $\lambda_{\{1\}} + \lambda_{\{1,2\}}$ regardless of the state of the pool
- Obligor 2 defaults with intensity $\lambda_{\{2\}} + \lambda_{\{1,2\}}$ regardless of the state of the pool

Markovian model of portfolio credit risk

General case: Obligor i defaults with intensity $\eta_i(t) = \sum_{Y \in \mathcal{Y}} \lambda_Y(t) \mathbf{1}_{\{i \in Y\}}$

$$\mathbb{P}(H^i_{t+dt} - H^i_t = 1 \mid \mathcal{F}_t) = \mathbb{P}(H^i_{t+dt} - H^i_t = 1 \mid H^i_t) = (1 - H^i_t)\eta_i(t)dt$$

- Each default indicator H^i, $i = 1, \ldots, n$ is a Markov process with respect to $\mathcal{F}$ (Property P2 is then satisfied)
- No contagion effect : Past defaults do not have any effect on intensities of surviving names

Markovian model of portfolio credit risk

The latter construction can be extended to the case of stochastic intensity functions:

$$\lambda_Y = \lambda_Y(t, \mathbf{X}_t)\,,\ Y \in \mathcal{Y}$$

where $\mathbf{X}_t = (X_t^1, \ldots, X_t^n)$ is a multivariate diffusion process:

$$dX_t^i = b_i(t, X_t^i)\,dt + \sigma_i(t, X_t^i)\,dW_t^i\,,\ i = 1, \ldots, n$$

- $\mathbf{W} = (W_t^1, \ldots, W_t^n)$: n-dimensional Brownian motion with correlation matrix $\varrho(t) = (\rho_{i,j}(t))_{1 \le i,j \le n}$
- b_i, σ_i are suitable drift and variance function-coefficients

Markovian model of portfolio credit risk

Markov property of the model

Let $\mathcal{F} = \mathcal{F}^{\mathbf{X},\mathbf{H}}$ be the natural filtration of $(\mathbf{X}, \mathbf{H})$. The process $(\mathbf{X}, \mathbf{H})$ is an $\mathcal{F}$-Markov process with generator $\mathcal{A}$ given by

$$\begin{aligned}
\mathcal{A}_t u(t, \mathbf{x}, \mathbf{k}) &= \sum_{1 \le i \le n} \Big(b_i(t, x_i) \partial_{x_i} u(t, \mathbf{x}, \mathbf{k}) + \frac{1}{2} \sigma_i^2(t, x_i) \partial^2_{x_i^2} u(t, \mathbf{x}, \mathbf{k}) \Big) \\
&+ \sum_{1 \le i < j \le n} \varrho_{i,j}(t) \sigma_i(t, x_i) \sigma_j(t, x_j) \partial^2_{x_i, x_j} u(t, \mathbf{x}, \mathbf{k}) \\
&+ \sum_{Y \in \mathcal{Y}} \lambda_Y(t, \mathbf{x}) \Big(u(t, \mathbf{x}, \mathbf{k}^Y) - u(t, \mathbf{x}, \mathbf{k}) \Big)
\end{aligned}$$

Markovian model of portfolio credit risk

The intensity of a jump of H^i from $H^i_{t-} = 0$ to 1 is given by:

$$\eta_i(t, \mathbf{X}_t) = \lambda_{\{i\}}(t, \mathbf{X}_t) + \sum_{k=1}^{m} \lambda_{I_k}(t, \mathbf{X}_t) \mathbf{1}_{\{i \in I_k\}}$$

Markov copula property

Under the following conditions

- $\lambda_{\{i\}}(t, \mathbf{x})$ only depends on $\mathbf{x} = (x_1, \ldots, x_n)$ through x_i, $i = 1, \ldots, n$
- $\lambda_{I_k}(t, \mathbf{x})$ does not depend on $\mathbf{x}$, $k = 1, \ldots, m$

for every $i = 1, \ldots, n$, the process (X^i, H^i) is an $\mathcal{F}$-Markov process admitting the following generator

$$\begin{aligned} \mathcal{A}^i_t u_i(t, x_i, k_i) \quad &= \quad b_i(t, x_i)\partial_{x_i} u_i(t, x_i, k_i) + \frac{1}{2}\sigma_i^2(t, x_i)\partial^2_{x_i^2} u_i(t, x_i, k_i) \\ &\quad + \eta_i(t, x_i)\big(u_i(t, x_i, 1) - u_i(t, x_i, k_i)\big) \end{aligned}$$

Practical implication: two-steps calibration procedure of single-name and multi-name products

Hedging CDO tranches using single-name CDS-s

Set of fundamental martingales (jump components)

- H_t^Z is the indicator process of simultaneous default of names in the set Z, for every subset Z of $\{1, \ldots, n\}$
- $Y_t = Y \cap \mathrm{supp}^c(\mathbf{H}_{t-})$ stands for the set of survivors of set Y right before t, for every pre-specified group $Y \in \mathcal{Y}$

Set of fundamental martingales

The process M^Z defined by

$$dM_t^Z := dH_t^Z - \ell_Z(t, \mathbf{X}_t, \mathbf{H}_{t-})dt$$

is a martingale with respect to $\mathcal{F}$, where the intensity function $\ell_Z(t, \mathbf{x}, \mathbf{k})$ is such that

$$\ell_Z(t, \mathbf{X}_t, \mathbf{H}_{t-}) = \sum_{Y \in \mathcal{Y};\, Y_t = Z} \lambda_Y(t, \mathbf{X}_t)$$

Hedging CDO tranches using single-name CDS-s

Itô formula

Given a "regular enough" function $u = u(t, \mathbf{x}, \mathbf{k})$, one has, for $t \in [0, T]$,

$$du(t, \mathbf{X}_t, \mathbf{H}_t) = \Big(\partial_t + \mathcal{A}_t\Big) u(t, \mathbf{X}_t, \mathbf{H}_t)dt + \nabla u(t, \mathbf{X}_t, \mathbf{H}_t)\sigma(t, \mathbf{X}_t) d\mathbf{W}_t$$
$$+ \sum_{Z \in \mathcal{Z}_t} \delta u^Z(t, \mathbf{X}_t, \mathbf{H}_{t-}) dM_t^Z$$

where

- $\sigma(t, \mathbf{x})$: diagonal matrix with diagonal $(\sigma_i(t, x_i))_{1 \le i \le n}$
- $\nabla u(t, \mathbf{x}, \mathbf{k}) = (\partial_{x_1} u(t, \mathbf{x}, \mathbf{k}), \ldots, \partial_{x_n} u(t, \mathbf{x}, \mathbf{k}))$
- $\delta u^Z(t, \mathbf{x}, \mathbf{k}) = u(t, \mathbf{x}, \mathbf{k}^Z) - u(t, \mathbf{x}, \mathbf{k})$
- $\mathcal{Z}_t = \{Y_t;\, Y \in \mathcal{Y}\} \setminus \emptyset$: set of all non-empty sets of survivors of sets Y in $\mathcal{Y}$ right before time t

Martingale dimension: $n + 2^n$

Hedging CDO tranches using single-name CDS-s

Price dynamics for single-name CDS-s (buy-protection position)

- T: contract maturity
- S_i: T-year contractual CDS-spread of obligor i
- $t_1 < \cdots < t_p = T$: premium payment dates, $h = t_j - t_{j-1}$ length between two premium payment dates (typically a quarter)
- R_i: recovery rate of obligor i

In the rest of the presentation, we will assume zero interest rates except for numerical results

Hedging CDO tranches using single-name CDS-s

Price dynamics for single-name CDS i

The price P^i and the cumulative value $\widehat{P}^i$ at time $t \in [0,T]$ of a single-name CDS on obligor i are given by

$$\begin{aligned} P_t^i &= \mathbf{1}_{\{\tau_i > t\}} v_i(t, X_t^i) \\ d\widehat{P}_t^i &= \mathbf{1}_{\{\tau_i > t\}} \partial_{x_i} v_i(t, X_t^i) \sigma_i(t, X_t^i) dW_t^i \\ &\qquad + \sum_{Z \in \mathcal{Z}_t} \mathbf{1}_{\{i \in Z\}} \left(1 - R_i - v_i(t, X_t^i)\right) dM_t^Z \end{aligned}$$

for a pre-default pricing function $v_i(t, x_i)$ such that

$$\mathbf{1}_{\{\tau_i > t\}} v_i(t, X_t^i) = \mathbb{E}[(1 - R_i)\mathbf{1}_{\{t < \tau_i \le T\}} - S_i h \sum_{t < t_j \le T} \mathbf{1}_{\{\tau_i > t_j\}} | \mathcal{F}_t]$$

Price dynamics for CDO tranche $[a, b]$ (buy-protection position)

- T: contract maturity
- a: attachement point, b: detachement point, $0 \leq a < b \leq 1$
- $S^{a,b}$: T-year contractual spread of CDO tranche $[a, b]$
- $t_1 < \cdots < t_p = T$: premium payment dates, $h = t_j - t_{j-1}$ length between two premium payment dates (typically a quarter)
- CDO tranche cash-flows are driven by the tranche loss process

$$L_t^{a,b} = L_{a,b}(\mathbf{H}_t) = (L_t - a)^+ - (L_t - b)^+$$

where

$$L_t = L_t(\mathbf{H}_t) = \frac{1}{n} \sum_{i=1}^{n} (1 - R_i) \mathbf{1}_{\{\tau_i \leq t\}}$$

is the credit loss process for the underlying portfolio

Hedging CDO tranches using single-name CDS-s

Price dynamics for CDO tranche $[a, b]$

The price Π and the cumulative value $\widehat{\Pi}$ at time $t \in [0, T]$ of a CDO-tranche $[a, b]$ are given by

$$\begin{aligned}
\Pi_t &= u(t, \mathbf{X}_t, \mathbf{H}_t) \\
d\widehat{\Pi}_t &= \nabla u(t, \mathbf{X}_t, \mathbf{H}_t)\sigma(t, \mathbf{X}_t) d\mathbf{W}_t \\
&\qquad + \sum_{Z \in \mathcal{Z}_t} \Big(L_{a,b}(\mathbf{H}^Z_{t-}) - L_{a,b}(\mathbf{H}_{t-}) + \delta u^Z(t, \mathbf{X}_t, \mathbf{H}_{t-}) \Big)\, dM^Z_t
\end{aligned}$$

for a pricing function $u(t, \mathbf{x}, \mathbf{k})$ such that

$$u(t, \mathbf{X}_t, \mathbf{H}_t) = \mathbb{E}\Big[L^{a,b}_T - L^{a,b}_t - S^{a,b}\, h \sum_{t < t_j \le T} \Big(b - a - L^{a,b}_{t_j} \Big) \,\Big|\, \mathcal{F}_t \Big]$$

The pricing function $u(t, \mathbf{x}, \mathbf{k})$ solves a very large system of Kolmogorov pde

Hedging CDO tranches using single-name CDS-s

Hedging portfolio: first d single-name CDS-s and the savings account

The vector of cumulative values $\widehat{\mathbf{P}} = (\widehat{P}^1, \ldots, \widehat{P}^d)^{\mathsf{T}}$ associated with the first d CDS-s has the following dynamics:

$$d\widehat{\mathbf{P}}_t = \nabla \mathbf{v}(t, \mathbf{X}_t, \mathbf{H}_t)\sigma(t, \mathbf{X}_t)d\mathbf{W}_t + \sum_{Z \in \mathcal{Z}_t} \Delta \mathbf{v}^Z(t, \mathbf{X}_t, \mathbf{H}_{t-})dM_t^Z$$

where

- $\nabla \mathbf{v}$ is a $d \times n$-matrix such that $\nabla \mathbf{v}(t, \mathbf{x}, \mathbf{k})_i^j = \mathbf{1}_{\{k_j = 0\}} \partial_{x_j} v_i(t, x_i)$, for every $1 \leq i \leq d$ and $1 \leq j \leq n$
- $\Delta \mathbf{v}^Z(t, \mathbf{x}, \mathbf{k})$ is a d-dimensional column vector equal to

$$\left(\mathbf{1}_{\{1 \in Z,\, k_1 = 0\}} \left((1 - R_1) - v_1(t, x_1)\right), \ldots, \mathbf{1}_{\{d \in Z,\, k_d = 0\}} \left((1 - R_d) - v_d(t, x_d)\right)\right)^{\mathsf{T}}$$

Hedging CDO tranches using single-name CDS-s

Tracking error: Process (e_t) such that $e_0 = 0$ and for $t \in [0, T]$:

$$\begin{aligned} de_t &= d\widehat{\Pi}_t - \zeta_t d\widehat{\mathbf{P}}_t \\ &= \Big(\nabla u(t, \mathbf{X}_t, \mathbf{H}_t) - \zeta_t \nabla \mathbf{v}(t, \mathbf{X}_t, \mathbf{H}_t)\Big)\sigma(t, \mathbf{X}_t) d\mathbf{W}_t \\ &\quad + \sum_{Z \in \mathcal{Z}_t} \Big(\Delta u^Z(t, \mathbf{X}_t, \mathbf{H}_{t-}) - \zeta_t \Delta \mathbf{v}^Z(t, \mathbf{X}_t, \mathbf{H}_{t-})\Big) dM_t^Z \end{aligned}$$

where

- $\zeta_t = (\zeta_t^1, \ldots, \zeta_t^d)$ gives the positions held at time t in CDS $1, \ldots, d$
- $\nabla u(t, \mathbf{x}, \mathbf{k}) = (\partial_{x_1} u(t, \mathbf{x}, \mathbf{k}), \ldots, \partial_{x_n} u(t, \mathbf{x}, \mathbf{k}))$
- $\Delta u^Z(t, \mathbf{x}, \mathbf{k}) = \delta^Z u(t, \mathbf{x}, \mathbf{k}) + L_{a,b}(\mathbf{k}^Z) - L_{a,b}(\mathbf{k})$

Hedging CDO tranches using single-name CDS-s

Min-variance hedging strategies

The min-variance hedging strategy ζ for the CDO-tranche $[a, b]$ is

$$\zeta_t = \frac{d\langle \widehat{\Pi}, \widehat{\mathbf{P}} \rangle_t}{dt} \left(\frac{d\langle \widehat{\mathbf{P}} \rangle_t}{dt} \right)^{-1} = \zeta(t, \mathbf{X}_t, \mathbf{H}_{t-})$$

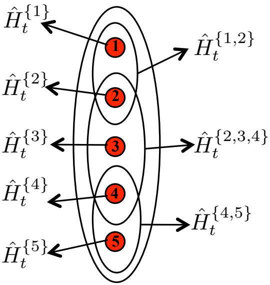
where $\zeta = (u, \mathbf{v})(\mathbf{v}, \mathbf{v})^{-1}$, with

$$(u, \mathbf{v}) = (\nabla u)\sigma^2(\nabla \mathbf{v})^{\mathsf{T}} + \sum_{Y \in \mathcal{Y}} \lambda_Y \Delta u^Y (\Delta \mathbf{v}^Y)^{\mathsf{T}}$$

$$(\mathbf{v}, \mathbf{v}) = (\nabla \mathbf{v})\sigma^2(\nabla \mathbf{v})^{\mathsf{T}} + \sum_{Y \in \mathcal{Y}} \lambda_Y \Delta \mathbf{v}^Y (\Delta \mathbf{v}^Y)^{\mathsf{T}}$$

Common-Shock Model Interpretation

Example: $n = 5$ and $\mathcal{Y} = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{4,5\},\{2,3,4\},\{1,2\}\}$.



$$\hat{H}_t^1 := \max\left\{\hat{H}_t^{\{1\}}, \hat{H}_t^{\{1,2\}}\right\}$$

$$\hat{H}_t^2 := \max\left\{\hat{H}_t^{\{2\}}, \hat{H}_t^{\{1,2\}}, \hat{H}_t^{\{2,3,4\}}\right\}$$

$$\hat{H}_t^3 := \max\left\{\hat{H}_t^{\{3\}}, \hat{H}_t^{\{2,3,4\}}\right\}$$

$$\hat{H}_t^4 := \max\left\{\hat{H}_t^{\{4\}}, \hat{H}_t^{\{2,3,4\}}, \hat{H}_t^{\{4,5\}}\right\}$$

$$\hat{H}_t^5 := \max\left\{\hat{H}_t^{\{5\}}, \hat{H}_t^{\{4,5\}}\right\}$$

General case: In the common-shock model, individual default indicators are such that

$$\widehat{H}_t^i := \max\left\{\widehat{H}_t^Y,\, Y \in \mathcal{Y},\, i \in Y\right\}$$

where $\widehat{H}_t^Y$, $Y \in \mathcal{Y}$ are independent $\{0,1\}$–point processes with intensity λ_Y

Common-Shock Model Interpretation

In the rest of the presentation, we consider no spread risk $\mathcal{F}_t = \mathcal{F}_t^{\mathbf{H}}$

Main result

- $\widehat{\tau}_i := \inf\left\{t \geq 0 \mid \widehat{H}_t^i = 1\right\}$, $i = 1, \ldots, n$: default times in the common-shock model
- $\tau_i := \inf\left\{t \geq 0 \mid H_t^i = 1\right\}$, $i = 1, \ldots, n$: default times in the Markovian model

Proposition

For all $t_1, \ldots, t_n \geq 0$, the following relation holds

$$\mathbb{P}\left(\widehat{\tau}_1 > t_1, \ldots, \widehat{\tau}_n > t_n\right) = \mathbb{P}\left(\tau_1 > t_1, \ldots, \tau_n > t_n\right)$$

Common-Shock Model Interpretation

Main result ($\mathcal{F}_t$–conditional version)

- supp$(\mathbf{H}_t)$: set of all defaulted names at time t
- $\mathcal{Y}_t = \{Y \in \mathcal{Y}\,;Y \nsubseteq \text{supp}(\mathbf{H}_t)\}$: set of pre-specified groups that contain at least one alive obligor
- $\widehat{H}^i_t := \max\left\{\widehat{H}^Y_t,\, Y \in \mathcal{Y}_t,\, i \in Y\right\}$: individual default processes in the $\mathcal{F}_t$–related common-shock model
- $\widehat{\tau}_i(t) := \inf\left\{\theta \geq t \mid \widehat{H}^i_\theta = 1\right\}$, $i \in \text{supp}^c(\mathbf{H}_t)$: default times of surviving names in the $\mathcal{F}_t$–related common-shock model
- $\tau_i := \inf\left\{\theta \geq t \mid H^i_\theta = 1\right\}$, $i \in \text{supp}^c(\mathbf{H}_t)$: default times of surviving names in the $\mathcal{F}_t$–conditional Markovian model

Proposition

Let Z be a subset of $\{1, \ldots, n\}$. For every $t_1, \ldots, t_n \geq t$, one has on the set $\{Z = \text{supp}^c(\mathbf{H}_t)\}$

$$\mathbb{P}\left(\widehat{\tau}_i(t) > t_i,\, i \in Z\right) = \mathbb{P}\left(\tau_i > t_i,\, i \in \text{supp}^c(\mathbf{H}_t) \mid \mathcal{F}_t\right)$$

Common-Shock Model Interpretation

Calibration of marginal default intensities on single-name CDS-s

- Individual shocks + Common shocks: $\mathcal{Y} = \{\{1\}, \ldots, \{n\}, I_1, \ldots, I_m\}$
- Price at time $t = 0$ of CDS i can be expressed as a function of $\mathbb{E}\left[H_t^i\right]$, $t = 0, \ldots, T$

$$\mathbb{E}\left[H_t^i\right] = \mathbb{P}(\tau_i > t) = 1 - \exp\left(-\int_0^t \eta_i(u)du\right)$$

where

$$\eta_i(u) = \lambda_{\{i\}}(u) + \sum_{k=1}^{m} \lambda_{I_k}(u)\mathbf{1}_{\{i \in I_k\}}$$

- Marginal default intensities η_i, $i = 1, \ldots, n$, can be calibrated on single-name CDS curves using a bootstrap procedure

Common-Shock Model Interpretation

Calibration of common-shocks intensities on CDO tranches

- Pricing of CDO tranches only involves marginal loss distributions
- Thanks to the common-shock model interpretation:

$$L_t = \frac{1}{n}\sum_{i=1}^{n}(1-R_i)H_t^i \stackrel{d}{=} \frac{1}{n}\sum_{i=1}^{n}(1-R_i)\widehat{H}_t^i$$

- Conditionally on $\left(\widehat{H}_t^{I_1},\dots,\widehat{H}_t^{I_m}\right)$, $\widehat{H}_t^1,\dots,\widehat{H}_t^n$ are independent Bernoulli's with random parameters

$$p_t^i = \begin{cases} 1 & i \in \cup_{k=1}^m \{I_k\,;\,\widehat{H}_t^{I_k}=1\} \\ 1-\exp\left(-\int_0^t \lambda_{\{i\}}(u)du\right) & \text{else} \end{cases}$$

where

$$\lambda_{\{i\}}(u) = \eta_i(u) - \sum_{k=1}^{m}\lambda_{I_k}(u)\mathbf{1}_{\{i\in I_k\}} \geq 0$$

Common-Shock Model Interpretation

Fast convolution-recursion procedure for computing loss distribution

- Let $N_t^{(k)} = \sum_{i=1}^{k} \widehat{H}_t^i$, $k = 1, \ldots, n$
- Let $q_t^{(k)}(i) = \mathbb{P}\left(N_t^{(k)} = i \mid \widehat{H}_t^{I_1}, \ldots, \widehat{H}_t^{I_m}\right)$, $i = 0, \ldots, k$
- The following recursion procedure can be used to compute the conditional loss distribution starting from $k = 0$ and $q_t^{(0)}(0) = 1$

$$\begin{cases} q_t^{(k+1)}(0) = (1 - p_t^{k+1}) \cdot q_t^{(k)}(0) \\ q_t^{(k+1)}(i) = p_t^{k+1} \cdot q_t^{(k)}(i-1) + (1 - p_t^{k+1}) \cdot q_t^{(k)}(i), \quad i = 1, \ldots, k \\ q_t^{(k+1)}(k+1) = p_t^{k+1} \cdot q_t^{(k)}(k) \end{cases}$$

- This gives the time-t conditional distribution $q_t^{(n)}$ of the total number of defaults $N_t := N_t^{(n)} = \sum_{i=1}^{n} \widehat{H}_t^i$

Common-Shock Model Interpretation

Fast convolution-recursion procedure for computing loss distribution

- The computation of the unconditional loss distribution involves a summation over 2^m terms, i.e., as many terms as possible states of the vector $(\widehat{H}_t^{I_1}, \dots, \widehat{H}_t^{I_m})$
- In the case where the groups are nested, i.e., $I_1 \subset \cdots \subset I_m$, the collection of events $(A_t^k)_{k=0,\dots,m}$ defined by

$$\begin{cases} A_t^0 = \left\{\widehat{H}_t^{I_1} = 0, \dots, \widehat{H}_t^{I_m} = 0\right\} \\ A_t^k = \left\{\widehat{H}_t^{I_k} = 1, \widehat{H}_t^{I_{k+1}} = 0, \dots, \widehat{H}_t^{I_m} = 0\right\}, \quad k = 1, \dots, m-1 \\ A_t^m = \left\{\widehat{H}_t^{I_m} = 1\right\} \end{cases}$$

forms a partition of Ω.

Common-Shock Model Interpretation

Fast convolution-recursion procedure for computing loss distribution

- Since A_t^k, $k = 0, \dots, m$ are disjoint events:

$$\mathbb{P}(N_t = i) = \sum_{k=1}^{m} \mathbb{P}(N_t = i \mid A_t^k)\mathbb{P}(A_t^k), \;\; i = 0, \dots, n$$

- $\mathbb{P}(N_t = i \mid A_t^k)$ can be computed thanks to the previous recursion procedure using the fact that $\widehat{H}_t^1, \dots, \widehat{H}_t^n$ are conditionally independent Bernoulli's given A_t^k, for every $k = 0, \dots, m$

- As $\widehat{H}_t^{I_1}, \dots, \widehat{H}_t^{I_m}$ are independent rv, the probability of the event A_t^k satisfies

$$\mathbb{P}(A_t^k) = \left(1 - \exp\left(-\int_0^t \lambda_{I_k}(u)du\right)\right) \prod_{j=k+1}^{m} \exp\left(-\int_0^t \lambda_{I_j}(u)du\right)$$

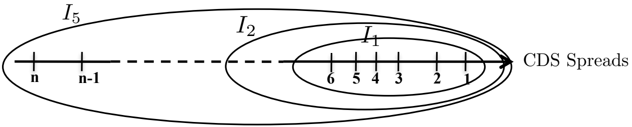
Calibration on CDX index

Data set: 5-year CDX North-America IG index on 20 December 2007

- Quoted spreads at different pillars of the $n = 125$ index constituents
- Quoted spreads of standard tranches [0,3], [3,7], [7,10], [10,15], [15,30]

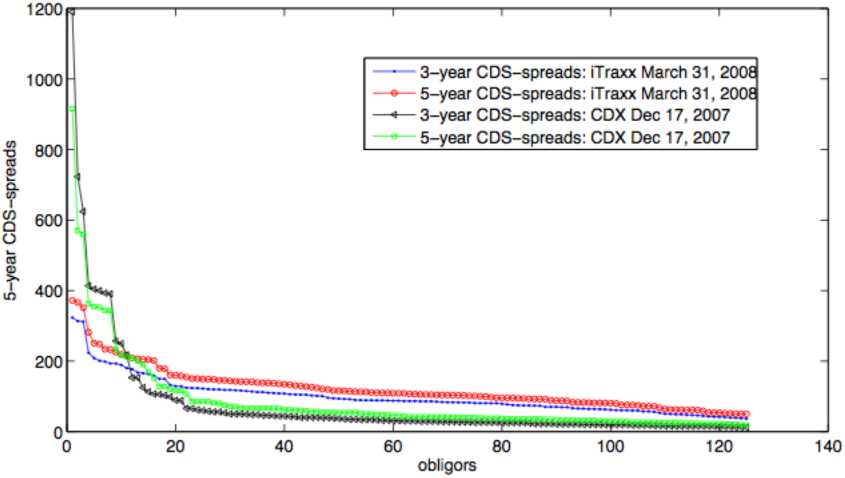
Model specification:

- Names are labelled with respect to decreasing level of spreads



- $m = 5$ groups $I_1 \subset \cdots \subset I_5$ such that $I_1 = \{1, \ldots, 6\}$, $I_2 = \{1, \ldots, 19\}$, $I_3 = \{1, \ldots, 25\}$, $I_4 = \{1, \ldots, 61\}$, $I_5 = \{1, \ldots, 125\}$
- Piecewise-constant intensities $\lambda_{\{1\}}, \ldots, \lambda_{\{125\}}$, $\lambda_{I_1}, \ldots, \lambda_{I_5}$ with grid points corresponding to CDS pillars
- Homogeneous and constant recovery rates: 40%
- Constant short-term interest rate: 3%

Calibration on CDX index



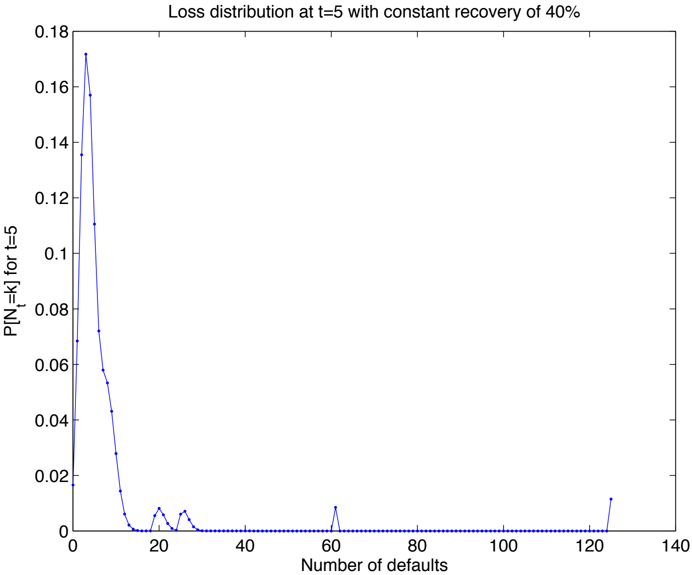
Calibration on CDX index

Calibration results:

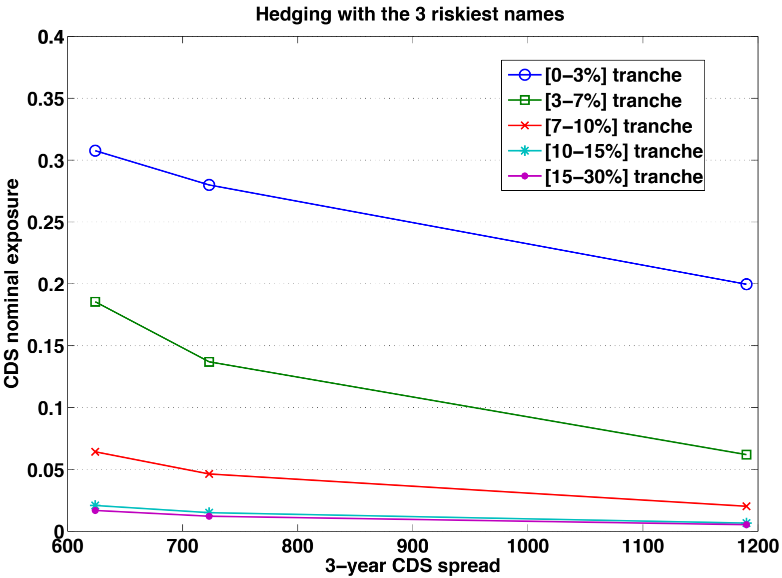
Tranche	[0,3]	[3,7]	[7,10]	[10,15]	[15,30]
Model spread in bps	48.0701	254.0000	124.0000	61.0000	38.9390
Market spread in bps	48.0700	254.0000	124.0000	61.0000	41.0000
Abs. Err. in bps	0.0001	0.0000	0.0000	0.0000	2.0610
% Rel. Err.	0.0001	0.0000	0.0000	0.0000	5.0269

- Names in the set $I_5 \setminus I_4$ are excluded from the calibration constraints (they can only default within the Armageddon shock I_5)

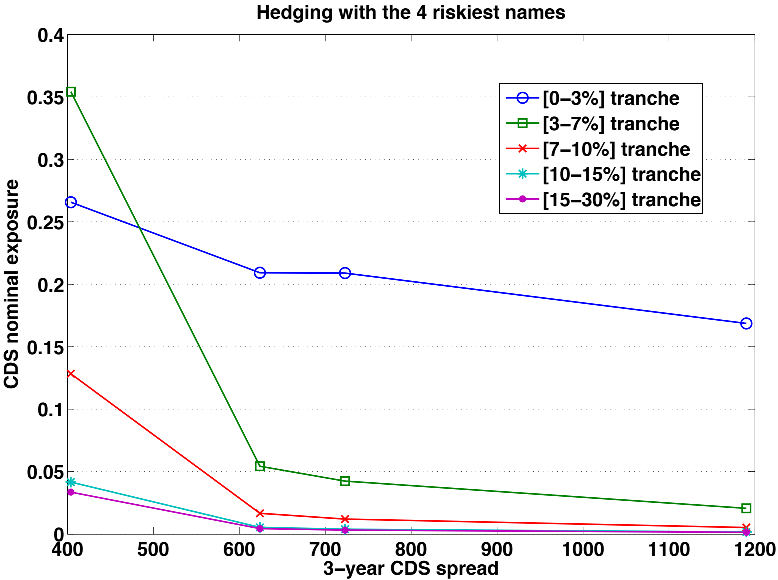
Implied 5-year loss distribution:



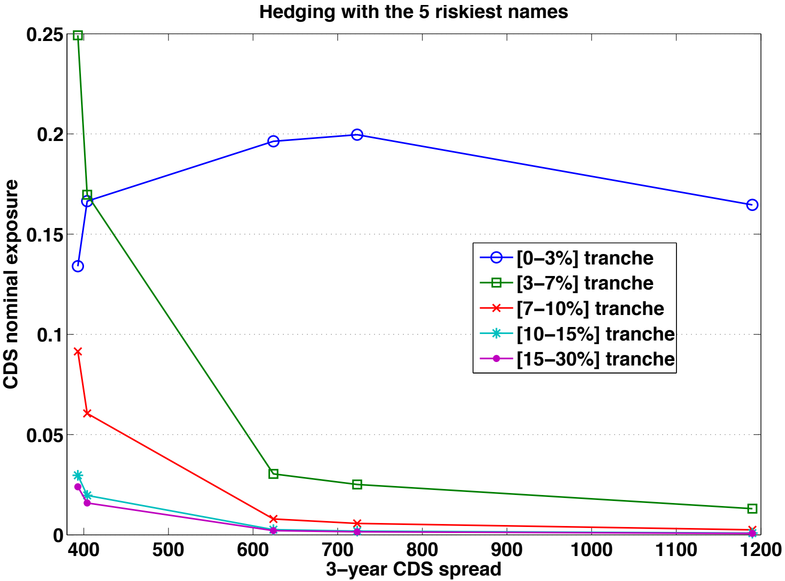
Min-variance hedging strategies



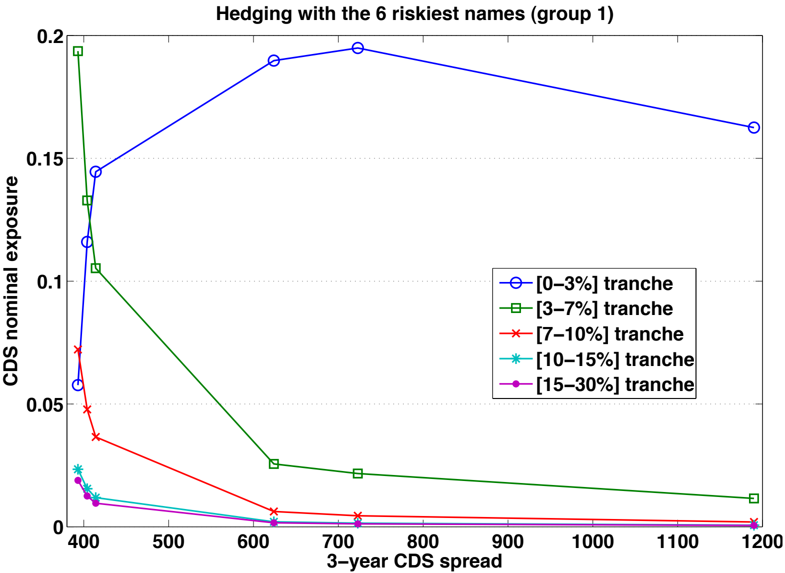
Min-variance hedging strategies



Min-variance hedging strategies



Min-variance hedging strategies



Thank you for your attention!

References

BIELECKI, T.R., VIDOZZI, A. AND VIDOZZI, L.: A Markov Copulae Approach to Pricing and Hedging of Credit Index Derivatives and Ratings Triggered Step–Up Bonds, *J. of Credit Risk*, 2008.

BRIGO, D., PALLAVICINI, A., TORRENTIAL, R.: Calibration of CDO Tranches with the Dynamical Generalized-Poisson Loss Model. *Working Paper*, 2006.

ELOUERKHAOUI, Y.: Pricing and Hedging in a Dynamic Credit Model. *International Journal of Theoretical and Applied Finance*, Vol. 10, Issue 4, 703–731, 2007.

LINDSKOG, F. AND MCNEIL, A. J.: Common Poisson Shock Models: Applications to Insurance and Credit Risk Modelling. *ASTIN Bulletin*, 33(2), 209-238, 2003