An Extension of Davis and Lo’s Contagion Model

Areski Cousin
Joint work with Diana Dorobantu and Didier Rullière

ISFA, University of Lyon


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Empirical studies on contagion mechanisms

- Das, Duffie, Kapadia, Saita (2007): Conditional independence assumption with no contagion effect is rejected by historical default data. The conditional independence assumption is not enough to fully capture the observed clustering in default events.

- Boissay (2006), Jorion and Zhang (2007, 2009) analyze the mechanism of default propagation and provide financial evidence of chain reactions or dominos effects.

Need for a dynamic model with defaults dependencies and contagion

- Default risks may be connected to underlying macro-economic factors
- Contagion mechanisms
- Chain reactions and evolution over time
Some contagion models in the credit risk field

- Intensities depending on defaults: Jarrow and Yu (2001), Yu (2007)
- Copula: Schönbucher and Schubert (2001)

In the spirit of Davis and Lo’s contagion model

- First models: Davis and Lo (2001)
- We propose a multiperiod extension of Davis and Lo’s contagion model.
Modeling of credit contagion for a pool of defaultable entities

- One-period model
- Credit references may default either directly or as a consequence of a contagion effect

Example: Portfolio with 5 credit references over one period

- No direct default ($X_1=0$)
- Direct default ($X_2=1$)
- Contagion ($Y_{23}=1$)
- No contagion ($Y_{24}=0$)
Davis and Lo’s contagion model

One-period contagion model

- \( n \) : number of credit references,
- \( X_i \): direct default indicator of name \( i \) (i.e. \( X_i = 1 \) if \( i \) defaults directly, \( X_i = 0 \) otherwise),
- \( Y_{ji} = 1 \) if the contagion link is activated from name \( j \) to name \( i \), \( Y_{ji} = 0 \) otherwise.
- \( C_i \): indirect default indicator of name \( i \),
- \( Z_i \): resulting default indicator (direct or indirect) such that :

\[
Z_i = X_i + (1 - X_i)C_i
\]

where :

\[
C_i = \prod \left\{ \begin{array}{l}
1 \\
\text{at least one } x_j Y_{ji}=1, j=1,...,n
\end{array} \right.
\]

\[
= 1 - \prod_{j \neq i} (1 - X_j Y_{ji})
\]
$N = \sum_{i=1}^{n} Z_i$ : total number of defaults

**Distribution of total number of defaults (Davis and Lo)**

\[
P[N = k] = C_n^k p^k (1 - p)^{n-k} (1 - q)^{k(n-k)} + \\
C_n^k \sum_{i=1}^{k-1} C_i^k p^i (1 - p)^{n-i} (1 - (1 - q)^i)^{k-i} (1 - q)^{i(n-k)}.
\]

**Under the assumptions :**

- Direct defaults $X_i, i = 1, \ldots, n$ : iid Bernoulli with parameter $p$
- Contagion links $Y_{ij}, i, j = 1, \ldots, n$ : iid Bernoulli with parameter $q$
- One contagion link alone may trigger an indirect default
- Infected entities cannot contaminate others (no chain-reaction effect)
**Extension of Davis and Lo’s contagion model**

**Dominos Effect**

- The model becomes a multiperiod model
- One can choose the set of entities likely to contaminate others
- Some iid assumptions are released

![Diagram showing the Dominos Effect](image-url)
Contagion incidence on indirect default

- One can change the number of contagion links required to cause a default (here two contaminations required)
Multi-period contagion model: \( t = 1, 2, \ldots, T \), in period \([t - 1, t]\):

- \( n \): number of credit references,
- \( X_t^i \): direct default indicator of entity \( i \),
- \( Y_t^{ji} \): contagion links are Bernoulli random variables such that \( Y_t^{ji} = 1 \) if entity \( j \) may infect entity \( i \),
- \( Z_t^i \): resulting default indicator (direct or indirect) such that:
  \[
  Z_t^i = Z_{t-1}^i + (1 - Z_{t-1}^i)[X_t^i + (1 - X_t^i)C_t^i]
  \]
- \( C_t^i = f \left( \sum_{j \in F_t} Y_t^{ji} \right) \): indirect default indicator of name \( i \),
- \( F_t \) is the set of names that are likely to infect other names between \( t \) and \( t + 1 \)
- \( f \) is a contamination trigger function, for example \( f = \mathbb{1}_{x \geq 1} \) (Davis and Lo) or \( f = \mathbb{1}_{x \geq 2} \)
Extension of Davis and Lo’s contagion model

\[ N_t = \sum_{i=1}^{n} Z_t^i : \text{total number of defaults at time } t \]

**Main result**

\[
P[N_t = r] = \sum_{k=0}^{r} P[N_{t-1} = k] C_{r-k}^{r-k} \sum_{\gamma=0}^{n-k-\gamma} C_{\gamma}^{\gamma} \cdot \sum_{\alpha=0}^{n-k-\gamma} C_{n-k-\gamma+\alpha}^{\alpha} \mu \gamma + \alpha, t \sum_{j=0}^{n-r} C_{n-r}^{j} (-1)^{j+\alpha} \xi_j + r - k - \gamma, t(\gamma).
\]

**Under the assumptions:**

- Direct defaults \(X_t^i, i = 1, \ldots, n\) are conditionally independent Bernoulli r.v. with the same marginal distribution and \(X_t = (X_t^1, \ldots, X_t^n)\), \(t = 1, \ldots, T\) are independent vectors.

- Contagion links \(Y_t^{ji}, i, j = 1, \ldots, n\) are conditionally independent Bernoulli r.v. with the same marginal distribution and \(Y_t = (Y_t^{ji})_{1 \leq i, j \leq n}\), \(t = 1, \ldots, T\) are independent vectors.

- \((X_t)_{t=1, \ldots, T}\) and \((Y_t)_{t=1, \ldots, T}\) are independent.
Calibration on 5-years iTraxx tranche quotes

- Cash-flows of CDO tranches driven by the aggregate loss process (in %)

\[ L_t = \frac{1}{n} \sum_{i=1}^{n} (1 - R_i)Z_t^i \]

where \( R_i \) is the recovery rate associated with name \( i \).

- if \( R_i = R \) for any \( i = 1, \ldots, n \)

\[ L_t = \frac{1}{n} (1 - R) \cdot N_t \]
We restrict ourselves to the case where for all $t$:

- Direct defaults $X_t^i \sim \text{Bernoulli}(\Theta)$ where $\Theta \sim \text{Beta}$, $\mathbb{E}[\Theta] = p$ and $\text{Var}(\Theta) = \sigma^2$, $i = 1, \ldots, n$
- Contagion links $Y_{t}^{ij}$ are iid $Y_{t}^{ij} \sim \text{Bernoulli}(q)$, $i, j = 1, \ldots, n$
- Only one default is required to trigger a default by contagion

Moreover

- $n = 125$, $r = 3\%$ (short-term interest rate)
- Recovery rate $R = 40\%$
- Computation of CDO tranche price only requires marginal loss distributions at several time horizons
Least square calibration procedure: Find $\alpha^* = (p^*, \sigma^*, q^*, R^*)$ which minimizes:

$$RMSE(\alpha) = \sqrt{\frac{1}{6} \sum_{i=1}^{6} \left( \frac{\tilde{s}_i - s_i(\alpha)}{\tilde{s}_i} \right)^2}.$$

where

<table>
<thead>
<tr>
<th>Market prices</th>
<th>$\tilde{s}_1$</th>
<th>$\tilde{s}_2$</th>
<th>$\tilde{s}_3$</th>
<th>$\tilde{s}_4$</th>
<th>$\tilde{s}_5$</th>
<th>$\tilde{s}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model prices</td>
<td>$s_1(\alpha)$</td>
<td>$s_2(\alpha)$</td>
<td>$s_3(\alpha)$</td>
<td>$s_4(\alpha)$</td>
<td>$s_5(\alpha)$</td>
<td>$s_0(\alpha)$</td>
</tr>
</tbody>
</table>
To improve the results we consider:

- One additional external contagious entity

<table>
<thead>
<tr>
<th>Date</th>
<th>0%-3%</th>
<th>3%-6%</th>
<th>6%-9%</th>
<th>9%-12%</th>
<th>12%-20%</th>
<th>Index</th>
<th>RMSE</th>
</tr>
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<tr>
<td>31 Jan 2008</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Market</td>
<td>31</td>
<td>317</td>
<td>212</td>
<td>140</td>
<td>74</td>
<td>77</td>
<td>-</td>
</tr>
<tr>
<td>Model</td>
<td>32</td>
<td>328</td>
<td>204</td>
<td>142</td>
<td>77</td>
<td>64</td>
<td>7.5</td>
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<tr>
<td>1st Mar 2007</td>
<td>10</td>
<td>46</td>
<td>13</td>
<td>6</td>
<td>2</td>
<td>23</td>
<td>-</td>
</tr>
<tr>
<td>Market</td>
<td>10</td>
<td>37</td>
<td>14</td>
<td>6</td>
<td>2</td>
<td>21</td>
<td>9.2</td>
</tr>
<tr>
<td>Model</td>
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<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Table: iTraxx Europe main market and model spreads (in bp) and the corresponding root mean square errors. The [0%-3%] spread is quoted in %. All maturities are for five years.
corresponding optimal parameters (on quarterly periods)

<table>
<thead>
<tr>
<th>Date</th>
<th>$p^*$</th>
<th>$\sigma_X^*$</th>
<th>$q^*$</th>
<th>$R^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>31 Jan 2008</td>
<td>0.0012</td>
<td>0.0151</td>
<td>0.0007</td>
<td>0.1964</td>
</tr>
<tr>
<td>1st Mar 2007</td>
<td>0.0001</td>
<td>0.0026</td>
<td>0.0005</td>
<td>0.1346</td>
</tr>
</tbody>
</table>

Table: Optimal parameters $\alpha^* = (p^*, \sigma_X^*, q^*, R^*)$. 
Conclusion

We propose a multi-period extension of Davis and Lo’s contagion model that accounts for

- possibly dominos or chain reaction effect
- flexible contagion mechanism (ex: more than one default required to trigger a contamination)

We provide a recursive formula for the distribution of the number of defaults at different time horizons

- especially when direct defaults and contagion events are conditionally independent

The multi-period setting is required to price synthetic CDO tranches

- The contagion parameter has a significant impact on the model ability to fit CDO tranche quotes
Thank you for your attention.
Similar kind of formulas hold when we have:

**finite-exchangeability**
- Direct defaults may be finite-exchangeable (does not imply conditional independence as infinite exchangeability, De Finetti’s Theorem does not apply here).

**non stationarity**
- Joint law for Direct defaults and for Contagion links may change over time.

**heterogeneity (with higher complexity)**
- Direct defaults may be dependent and heterogeneous, in a monoperiodic framework.
- Direct defaults may be dependent and heterogeneous, in a multiperiodic framework, but with an exponential complexity (need to consider all possible sets of remaining entities at time $t$).
Waring’s Formula - special case of Schuette-Nesbitt Formula

If $B^1, \ldots, B^n$ are $n$ dependent Bernoulli r.v. and $\Gamma \subset \{1, \ldots, n\}$ with cardinal $m$, then

$$P \left[ \sum_{i \in \Gamma} B^i = k \right] = \mathbb{1}_{k \leq m} C_m^k \sum_{j=0}^{m-k} C_{m-k}^j (-1)^j \mu_{j+k}(\Gamma).$$

with $\mu_k(\Gamma) = \frac{1}{C_m^k} \sum_{j_1<j_2<\ldots<j_k \atop j_1, \ldots, j_k \in \Gamma} P \left[ B^{j_1} = 1 \cap \ldots \cap B^{j_k} = 1 \right]$, $k \geq 1$,

coefficients $\mu_k$ may be simplified:

- if independence (without requiring iid): products
- if exchangeability: the sum vanishes

Here we are looking for:

- Directs defaults: $\sum_{j \in \Gamma} X^j_t$ as a function of some coefficients $\mu_{k,t}(\Gamma)$,
- Contagion links: $\sum_{j \in F_t} Y^{\sigma(j)}_t$ as a function of some coefficients $\lambda_{k,t}$,
- Indirects defaults: $\sum_{j=1}^{k} C^j_t$ as a function of some coefficients $\xi_{k,t}$,
Appendix I - probabilistic tools
### Infinite- exchangeability

A₁, A₂, ... sequence of exchangeable r.v. if for all n and for any permutation σ

\[ A_1, \ldots, A_n \overset{D}{=} A_{\sigma(1)}, \ldots, A_{\sigma(n)}, \]

### De Finetti’s Theorem

A₁, A₂, ... is a sequence of infinite-exchangeable Bernoulli r.v. iff there exist a r.v. \( \Theta \in [0, 1] \) such that, conditionally to \( \Theta \)

- A₁, A₂, ... is an iid sequence of Bernoulli r.v. with parameter \( \Theta \)
- Here, calculations given \( \Theta \) but difficulties to simplify
- De Finetti’s Theorem does not apply for finite-exchangeability
- Need for other tools
Appendix I - Probabilistic tools

If $N$ is a number of fulfilled events $B_i$, $i \in \Omega$,
A linear combination of $P[N = k]$ will be written:

**Schuette-Nesbitt formula**

$$\sum_{k \in \Omega} P[N = k]f(k) = \sum_{k \in \Omega} S_k \Delta^k f(0)$$

avec $S_k = \sum_{j_1 < \cdots < j_k} P[B_{j_1} \cap \cdots \cap B_{j_k}]$

$$\Delta f(k) = f(k + 1) - f(k), \text{ difference operator}$$

- events of kind $[N = k]$ given coefficients $S_k$.
- $S_k$ can be simplified with independence, without requiring i.i.d.
- $S_k$ can be simplified with exchangeability
- events of kind $[N = k]$ as simple as $[N = 0]$ or $[N \geq 1]$
Appendix I - Probabilistic tools

In the particular case where \( f(j) = \mathbb{1}_{j \equiv k}, j \in \Omega, \)

\[ Waring's\ formula \]

If \( X_1^t, \ldots, X_n^t \) are \( n \) dependent Bernoulli r.v. and \( \Gamma \subset \Omega \) with cardinal \( m, \)

\[
P \left[ \sum_{i \in \Gamma} X_i^t = k \right] = \mathbb{1}_{k \leq m} C_m^k \sum_{j=0}^{m-k} C_{m-k}^j (-1)^j \mu_{j+k, t}(\Gamma).\]

with

\[
\mu_{k, t}(\Gamma) = \frac{1}{C_{\text{card}(\Gamma)}^k} \sum_{j_1 < j_2 < \ldots < j_k} \sum_{j_1, \ldots, j_k \in \Gamma} P \left[ X_{j_1}^t = 1 \cap \ldots \cap X_{j_k}^t = 1 \right], \quad k \geq 1,
\]

\[
\mu_{0, t}(\Gamma) = 1 \text{ (even if } \Gamma = \emptyset).\]
Interest in life-insurance framework:
- independence assumptions
- but different ages and non identically distributed lifetimes

Interest for Davis and Lo extension:
- one would like $P[N = k]$
- on can change more easily iid assumptions
- is simplified with exchangeability assumptions
Appendix I - Probabilistic tools

Idea from so-called Waring’s formula

for non i.i.d. Bernoulli r.v. $A_1, \ldots, A_n$, one can get the law of $\sum_j A_j$ as a function of coefficients of kind

$$P [A_1 = 1 \cap \cdots \cap A_i = 1].$$

- If independence: these coefficients become products
- If exchangeability: these coefficients does only depend on the number of considered r.v.

Here we are looking for:

- **Directs defaults**: $\sum_{j \in \Gamma} X_t^j$ as a function of coefficients $\mu_{k,t}(\Gamma)$,
- **Contagion links**: $\sum_{j \in F_t} Y_t^{\sigma(j)}$ as a function of coefficients $\lambda_{k,t}$,
- **Indirects defaults**: $\sum_{j=1}^{k} C_t^j$ as a function of coefficients $\xi_{k,t}$,
Appendix II - Basic numerical illustration
we consider here that for all $t$,
- $X_t^i$ are exchangeables, Bernoulli with hidden parameter $\Theta_X$, $E[\Theta_X] = p = 0.1$, $V[\Theta_X]$ is given
- $Y_t^{ij}$ are exchangeables, Bernoulli with hidden parameter $\Theta_Y$, $E[\Theta_Y] = q = 0.2$, $V[\Theta_Y]$ is given
- hidden parameters are Beta distributed

We consider
- 10 entities ($n = 10$),
- 10 temporal units ($T = 10$),
- average direct default probability $p = 0.1$,
- average contagion link probability $q = 0.2$. 
We define 4 models with common parameters:

1. **model 1**: $\sigma_X = 0$, $\sigma_Y = 0$, $f(x) = \mathbb{1}_{x \geq 1}$ (i.i.d. case, one contagion link required).

2. **model 2**: $\sigma_X = 0$, $\sigma_Y = 0$, $f(x) = \mathbb{1}_{x \geq 2}$ (i.i.d. case, two contagion links required).

3. **model 3**: $\sigma_X = 0.2$, $\sigma_Y = 0.2$, $f(x) = \mathbb{1}_{x \geq 1}$ (exchangeable case, one contagion link required).

4. **model 4**: $\sigma_X = 0.2$, $\sigma_Y = 0.2$, $f(x) = \mathbb{1}_{x \geq 2}$ (exchangeable case, two contagion link required).
Evolution of $E[N_t]$ as a function of $t$. i.i.d. case dotted.
Evolution of $V[N_t]$ as a function of $t$. i.i.d. case dotted.
Evolution of $P[N_t \geq 6]$ as a function of $t$. i.i.d. case dotted.
Evolution of $P[N_t \geq 10]$ as a function of $t$. i.i.d. case dotted.
Appendix III - Some remarks

specificity of the model
- try to capture explicit microstructure of contagion
- contagion acts directly on random variables, not on probabilities
- one can say with certainty if default of entity $i$ is due to entity $j$
- acts in a complete graph

some limits of the model
- default rate depends on the number $n$ of entities
- contagions only within the considered portofolio
- numerical issues for large number $n$ of entities

some perspectives
- recursions to manage numerical issues
- contagions from outside the portofolio
- behavior when time tends to zero and $n$ becomes large
- asymptotic results – larger interconnected component
- recovery effects
- Heterogeneity via a small number of groups