A multivariate extension of VaR and CTE measures

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MAF 2012, Venice,

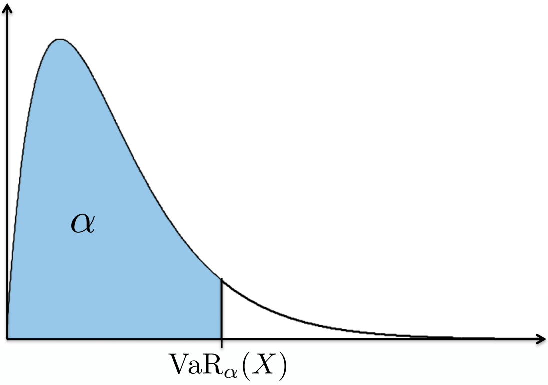
10 April 2012

- Financial institutions strongly rely on the Value-at-Risk paradigm to compute regulatory and economic capital

- Under Basel II or Solvency II, regulatory capital required for one particular financial institution does not take into account risks undertaken by the other institutions even if the latter are highly interconnected. (“micro-prudential regulation”)

- We propose in this work a multivariate extension of the classical VaR and CTE risk measures

Value-at-Risk measure



Given an univariate continuous and strictly monotonic loss distribution function F_X,

$$\mathrm{VaR}_\alpha(X) = Q_X(\alpha) = F_X^{-1}(\alpha), \quad \forall\, \alpha \in (0,1).$$

Shortcoming of VaR measure:

- VaR does not give any information on the severity of loss when larger than the VaR
- VaR is not a coherent risk measure (see Artzner, 1999)

`To overcome problems of VaR → Conditional-Tail-Expectation (CTE):`

$$CTE_\alpha(X) = \mathbb{E}[\, X \,|\, X \geq \mathrm{VaR}_\alpha(X) \,] = \mathbb{E}[\, X \,|\, X \geq Q_X(\alpha) \,],$$

Dependence and dimensional problems

Riskiness not only of the marginal distributions, but also of the joint distribution:

$$\rho: \quad \mathbf{X} := (X_1, \ldots, X_d) \mapsto \begin{pmatrix} \rho^1[\mathbf{X}] \\ \vdots \\ \rho^d[\mathbf{X}] \end{pmatrix} \in \mathbb{R}_+^d,$$

Risk measures essentially based on a "*distributional approach*" (i.e. we have to capture the information coming both from the marginal distributions and from the dependence structure).

Multivariate *Value-at-Risk* as quantile curve (Embrechts & Puccetti, 2006; Nappo & Spizzichino, 2009), i.e., the set

$$\partial L(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d \,:\, F(\mathbf{x}) = \alpha\}$$

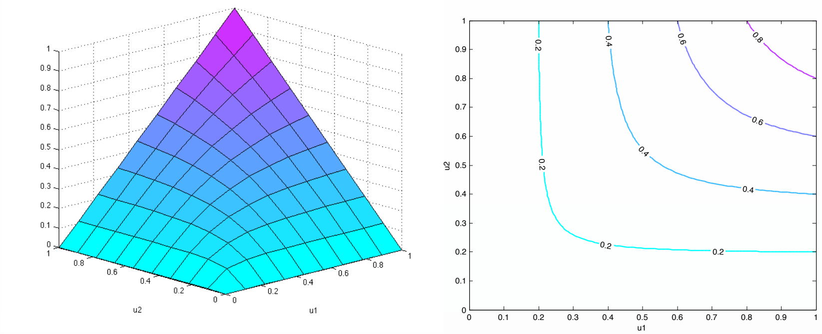


Figure: **left**: cdf of a Clayton copula with parameter 2, **right**: a set of associated quantile curves

A multivariate *Value-at-Risk* and *Conditional-Tail-Expectation*

Definition

Consider a random vector $\mathbf{X}$ with absolutely continuous cdf F. For $\alpha \in (0,1)$, we define:

$$\mathrm{VaR}_\alpha(\mathbf{X}) = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, \mathbf{X} \in \partial L(\alpha) \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, \mathbf{X} \in \partial L(\alpha) \,] \end{pmatrix} = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, F(\mathbf{X}) = \alpha \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, F(\mathbf{X}) = \alpha \,] \end{pmatrix},$$

$$\mathrm{CTE}_\alpha(\mathbf{X}) = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, \mathbf{X} \in L(\alpha) \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, \mathbf{X} \in L(\alpha) \,] \end{pmatrix} = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, F(\mathbf{X}) \geq \alpha \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, F(\mathbf{X}) \geq \alpha \,] \end{pmatrix},$$

where $\partial L(\alpha)$ is the α-level set of F and $L(\alpha)$ is the upper α-level set of F .

Example: Bivariate Archimedean copula case

$$\mathrm{VaR}^1_\alpha(X,Y) = \frac{\int_{Q_{\mathbf{X}}(\alpha)}^{\infty} x\, f_{(U,C(U,V))}(F_X(x),\alpha)\,\mathrm{d}x}{K'(\alpha)},$$

where $f_{(U,C(U,V))}$ is the density of the cdf $F_{(U,C(U,V))}$ given by

$$F_{(U,C(U,V))}(s,t) = t - \frac{\phi(t)}{\phi'(t)} + \frac{\phi(s)}{\phi'(t)}, \qquad \text{for } 0 < t < s < 1.$$

and K is the cdf of $C(U,V)$ (Kendall distribution)

Copula	θ	$\mathrm{VaR}^1_{\alpha,\theta}(X,Y)$
Clayton C_θ	$(-1,\infty)$	$\frac{\theta}{\theta-1}\frac{\alpha^\theta-\alpha}{\alpha^\theta-1}$
Counter-monotonic W	-1	$\frac{1+\alpha}{2}$
Independent Π	0	$\frac{\alpha-1}{\ln\alpha}$
Comonotonic M	∞	α

Example: Bivariate Clayton copula case

Remark: for Clayton $\frac{\partial \mathrm{VaR}^{\mathbf{1}}_{\alpha,\theta}}{\partial \alpha} \geq 0$ and $\frac{\partial \mathrm{VaR}^{\mathbf{1}}_{\alpha,\theta}}{\partial \theta} \leq 0$, for $\theta \geq -1$ and $\alpha \in (0,1)$.

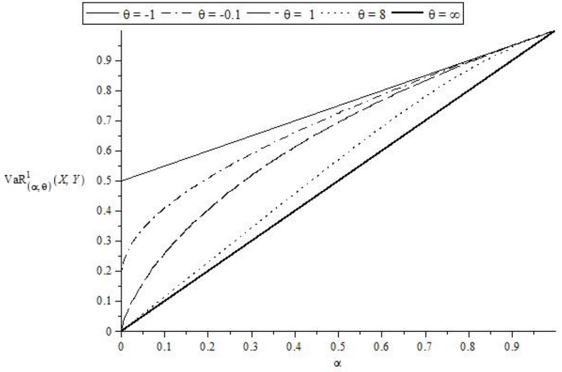


Figure: Behavior of $\mathrm{VaR}^1_{\alpha,\theta}(X,Y) = \mathrm{VaR}^2_{\alpha,\theta}(X,Y)$ with respect to risk level α for different values of dependence parameter θ. The random vector (X,Y) follows a Clayton copula distribution with parameter θ.

Example: Bivariate Clayton copula case

Copula	θ	$\mathrm{CTE}^1_{\alpha,\theta}(X,Y)$
Clayton C_θ	$(-1,\infty)$	$\frac{1}{2}\,\frac{\theta}{\theta-1}\,\frac{\theta-1-\alpha^2(1+\theta)+2\,\alpha^{1+\theta}}{\theta-\alpha(1+\theta)+\alpha^{1+\theta}}$
Counter-monotonic W	-1	$\frac{1}{4}\,\frac{1-\alpha^2+2\ln\alpha}{1-\alpha+\ln\alpha}$
Independent Π	0	$\frac{1}{2}\,\frac{(1-\alpha)^2}{1-\alpha+\alpha\ln\alpha}$
Comonotonic M	∞	$\frac{1+\alpha}{2}$

Table: $\mathrm{CTE}^1_{\alpha,\theta}(X,Y)$ for different copula dependence structures.

Interestingly, one can readily show that $\frac{\partial \mathrm{CTE}^1_{\alpha,\theta}}{\partial\alpha} \geq 0$ and $\frac{\partial \mathrm{CTE}^1_{\alpha,\theta}}{\partial\theta} \leq 0$, for $\theta \geq -1$ and $\alpha \in (0,1)$.

Example: Bivariate Frank copula case

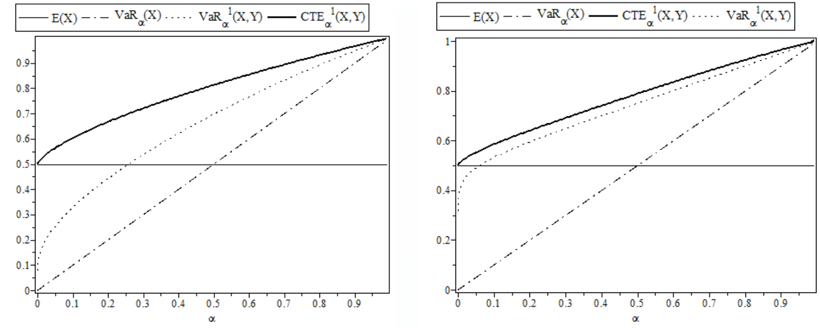


Figure: Frank copula with standard uniform marginals, parameter $\theta = 2$ (left), parameter $\theta = -10$ (right).

	$\text{VaR}_\alpha(\mathbf{X})$	$\text{CTE}_\alpha(\mathbf{X})$
Several (axiomatic) properties	Invariance properties ($\mathbf{c} \in \mathbb{R}_+^d$): • $\text{VaR}_\alpha(\mathbf{c}\,\mathbf{X}) = \mathbf{c}\,\text{VaR}_\alpha(\mathbf{X})$, • $\text{VaR}_\alpha(\mathbf{c}+\mathbf{X}) = \mathbf{c} + \text{VaR}_\alpha(\mathbf{X})$. Lower bounds: • $\text{VaR}^i_\alpha(\mathbf{X}) \geq \text{VaR}_\alpha(X_i)$, $\forall\, \alpha \in (0,1)$. Analytical closed-form formulas for $\text{VaR}_\alpha(\mathbf{X})$ and $\text{CTE}_\alpha(\mathbf{X})$.	Invariance properties ($\mathbf{c} \in \mathbb{R}_+^d$): • $\text{CTE}_\alpha(\mathbf{c}\,\mathbf{X}) = \mathbf{c}\,\text{CTE}_\alpha(\mathbf{X})$, • $\text{CTE}_\alpha(\mathbf{c}+\mathbf{X}) = \mathbf{c} + \text{CTE}_\alpha(\mathbf{X})$. Lower bounds: • $\text{CTE}^i_\alpha(\mathbf{X}) \geq \text{VaR}_\alpha(X_i)$, $\forall\, \alpha \in (0,1)$. Safety loading: • $\text{CTE}^i_\alpha(\mathbf{X}) \geq \mathbb{E}[X_i]$ • $\text{CTE}_0(\mathbf{X}) = \mathbb{E}[\mathbf{X}]$.
Risk level	$\text{VaR}^i_\alpha(\mathbf{X})$ is a non-decreasing function of α.	• $\text{CTE}^i_\alpha(\mathbf{X}) \geq \text{VaR}^i_\alpha(\mathbf{X})$, $\forall\, \alpha \in (0,1)$. • $\text{CTE}^i_\alpha(\mathbf{X})$ is a non-decreasing function of α.

✓ These two risk measures both satisfy the positive homogeneity and the translation invariance property (Artzner *et al.*, 1999).

✓ Comparison results between univariate risk measures and components of multivariate risk measures are provided.

✓ Change in risk level α.

	$\mathrm{VaR}_\alpha(\mathbf{X})$	$\mathrm{CTE}_\alpha(\mathbf{X})$
Dependence structure	Comonotonic case: • $\mathrm{VaR}^{i}_{\alpha}(\mathbf{X}) = \mathrm{VaR}_\alpha(X_i),\ \forall\,\alpha \in (0,1)$. For a fixed copula C and $X_i \overset{d}{=} Y_i$: • $\mathrm{VaR}^{i}_{\alpha}(\mathbf{X}) = \mathrm{VaR}^{i}_{\alpha}(\mathbf{Y}),\ \forall\,\alpha \in (0,1)$. For a fixed copula C and $X_i \preceq_{st} Y_i$: • $\mathrm{VaR}^{i}_{\alpha}(\mathbf{X}) \leq \mathrm{VaR}^{i}_{\alpha}(\mathbf{Y}),\ \forall\,\alpha \in (0,1)$.	Comonotonic case: • $\mathrm{CTE}^{i}_{\alpha}(\mathbf{X}) = \mathrm{CTE}_\alpha(X_i),\ \forall\,\alpha \in (0,1)$. For a fixed copula C and $X_i \overset{d}{=} Y_i$: • $\mathrm{CTE}^{i}_{\alpha}(\mathbf{X}) = \mathrm{CTE}^{i}_{\alpha}(\mathbf{Y}),\ \forall\,\alpha \in (0,1)$. For a fixed copula C and $X_i \preceq_{D} Y_i$: • $\mathrm{CTE}^{i}_{\alpha}(\mathbf{X}) \leq \mathrm{CTE}^{i}_{\alpha}(\mathbf{Y}),\ \forall\,\alpha \in (0,1)$.

✓ Change in marginal distributions and in dependence structure and by a change in risk level.

✓ Results turn to be consistent with existing properties on univariate risk measures.

✓ $\theta \leq \theta^* \Rightarrow \mathrm{VaR}^1_\alpha(X^*, Y^*) \leq \mathrm{VaR}^1_\alpha(X, Y)$ (Archimedean copula family).

Perspectives

A. Cousin, E. Di Bernardino, *A multivariate extension of Value-at-Risk and Conditional-Tail-Expectation*, submitted to *Journal of Multivariate Analysis* (2011), http://hal.archives-ouvertes.fr/hal-00638382/fr/.

- ✓ Comparisons of our multivariate CTE and VaR with existing multivariate generalizations of these measures, both theoretically and experimentally: applications on financial portfolios; micro-prudential versus macro-prudential approach, ...
- ✓ Our risk measures to the case of discrete distribution functions.

Thank you for your attention

Other multivariate risk measures in the litterature

Several multivariate generalizations of CTE. For $i = 1, \dots, d$

- $\mathrm{CTE}^{\mathrm{sum}}_{\alpha}(X_i) = \mathbb{E}[\, X_i \,|\, S > Q_S(\alpha)\,]$ where $S = X_1 + \cdots + X_d$
- $\mathrm{CTE}^{\mathrm{min}}_{\alpha}(X_i) = \mathbb{E}[\, X_i \,|\, X_{(1)} > Q_{X_{(1)}}(\alpha)\,]$ where $X_{(1)} = \min\{X_1, \dots, X_d\}$
- $\mathrm{CTE}^{\mathrm{max}}_{\alpha}(X_i) = \mathbb{E}[\, X_i \,|\, X_{(d)} > Q_{X_{(d)}}(\alpha)\,]$ where $X_{(d)} = \max\{X_1, \dots, X_d\}$

For Farlie-Gumbel-Morgenstern copula (Bargès *et al.*, 2009). For elliptic distribution functions (Landsman and Valdez, 2003). For phase-type distributions (Cai and Li, 2005).

- Inappropriate to measure risks with heterogeneous characteristics especially in an external risks problem