Dynamic hedging of synthetic CDO tranches

Areski Cousin
ISFA, Université Lyon 1

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Introduction

- In this presentation, we address the hedging issue of CDO tranches in a market model where pricing is connected to the cost of the hedge
- In credit risk market, models that connect pricing to the cost of the hedge have been studied quite lately
- Discrepancies with the interest rate or the equity derivative market
- Model to be presented is not new, require some stringent assumptions, but the hedging can be fully described in a dynamical way

Introduction

Presentation related to the papers :

- *Hedging default risks of CDOs in Markovian contagion models* (2008), Quantitative Finance, with Jean-Paul Laurent and Jean-David Fermanian
- *Delta-hedging correlation risk ?* (2010), submitted, with Stéphane Crépey and Yu Hang Kan

Contents

Default times

- n credit references
- τ_1,...,τ_n : default times defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$
- $N_t^i = 1_{\{\tau_i \leq t\}}$, $i = 1, \ldots, n$: default indicator processes
- $\mathbb{H}^i = (\mathcal{H}_t^i)_{t \geq 0}$, $\mathcal{H}_t^i = \sigma(N_s^i,\ s \leq t)$, $i = 1, \ldots, n$: natural filtration of N^i
- $\mathbb{H} = \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^n$: global filtration of default times

Default times

- No simultaneous defaults : $\mathbb{P}(\tau_i = \tau_j) = 0, \forall i \neq j$
- Default times admit $\mathbb{H}$-adapted default intensities
  - For any $i = 1, \ldots, n$, there exists a non-negative $\mathbb{H}$-adapted process $\alpha^{i,\mathbb{P}}$ such that
    $$M_t^{i,\mathbb{P}} := N_t^i - \int_0^t \alpha_s^{i,\mathbb{P}} ds$$
    is a $(\mathbb{P}, \mathbb{H})$-martingale.
  - $\alpha_t^{i,\mathbb{P}} = 0$ on the set $\{t > \tau_i\}$
  - $M^{i,\mathbb{P}}$, $i = 1, \ldots, n$ will be referred to as the fundamental martingales

Market Assumption

- Instantaneous digital CDS are traded on the names $i = 1, \dots, n$
- Instantaneous digital CDS on name i at time t is a stylized bilateral agreement
  - Offer credit protection on name i over the short period $[t, t + dt]$
  - Buyer of protection receives 1 monetary unit at default of name i
  - In exchange for a fee equal to $\alpha_t^i dt$

$$
\begin{array}{lll}
 & \nearrow & 1 - \alpha_t^i dt \text{ : default of } i \text{ between } t \text{ and } t + dt \\
0 & & \\
 & \searrow & -\alpha_t^i dt \text{ : survival of name } i \\
t & & t + dt
\end{array}
$$

- Cash-flow at time $t + dt$ (buy protection position) : $dN_t^i - \alpha_t^i dt$
- $\alpha_t^i = 0$ on the set $\{t > \tau_i\}$ (Contrat is worthless)

Market Assumption

- Credit spreads are driven by defaults : $\alpha^1, \ldots, \alpha^n$ are $\mathbb{H}$-adapted processes
- Payoff of a self-financed strategy

$$V_0 e^{rT} + \sum_{i=1}^{n} \int_0^T \delta_s^i e^{r(T-s)} \underbrace{\left(dN_s^i - \alpha_s^i ds\right)}_{\text{CDS cash-flow}}.$$

- r : default-free interest rate
- V_0 : initial investment
- δ^i, $i = 1, \ldots, n$, $\mathbb{H}$-predictable process

Hedging and martingale representation theorem

Theorem (Predictable representation theorem)

Let $A \in \mathcal{H}_T$ *be a* $\mathbb{P}$*-integrable random variable. Then, there exists* $\mathbb{H}$*-predictable processes* $\theta^i, i = 1, \dots, n$ *such that*

$$\begin{aligned} A &= \mathbb{E}_{\mathbb{P}}[A] + \sum_{i=1}^{n} \int_0^T \theta_s^i \left(dN_s^i - \alpha_s^{i,\mathbb{P}} ds \right) \\ &= \mathbb{E}_{\mathbb{P}}[A] + \sum_{i=1}^{n} \int_0^T \theta_s^i dM_s^{i,\mathbb{P}} \end{aligned}$$

and $\mathbb{E}_{\mathbb{P}} \left(\int_0^T |\theta_s^i| \alpha_s^{i,\mathbb{P}} ds \right) < \infty$.

Hedging and martingale representation theorem

Theorem (Predictable representation theorem)

Let $A \in \mathcal{H}_T$ *be a* $\mathbb{Q}$*-integrable random variable. Then, there exists* $\mathbb{H}$*-predictable processes* $\hat{\theta}^i, i = 1, \ldots, n$ *such that*

$$\begin{aligned} A &= \mathbb{E}_{\mathbb{Q}}[A] + \sum_{i=1}^{n} \int_0^T \hat{\theta}^i_s \underbrace{\left(dN^i_s - \alpha^i_s ds\right)}_{\text{CDS cash-flow}} \\ &= \mathbb{E}_{\mathbb{Q}}[A] + \sum_{i=1}^{n} \int_0^T \hat{\theta}^i_s dM^i_s \end{aligned}$$

and $\mathbb{E}_{\mathbb{Q}}\left(\int_0^T |\theta^i_s| \alpha^{i,\mathbb{P}}_s ds\right) < \infty.$

Hedging and martingale representation theorem

Building a change of probability measure

- Describe what happens to default intensities when the original probability is changed to an equivalent one
- From the PRT, any Radon-Nikodym density ζ (strictly positive $(\mathbb{P}, \mathbb{H})$-martingale with expectation equal to 1) can be written as

$$d\zeta_t = \zeta_{t-} \sum_{i=1}^{n} \pi_t^i dM_t^{i,\mathbb{P}}, \ \zeta_0 = 1$$

where π^i, $i = 1, \ldots, n$ are $\mathbb{H}$-predictable processes

Hedging and martingale representation theorem

- Conversely, the (unique) solution of the latter SDE is a local martingale (Doléans-Dade exponential)

$$\zeta_t = \exp\left(-\sum_{i=1}^{n}\int_0^t \pi_s^i \alpha_s^{i,\mathbb{P}} ds\right)\prod_{i=1}^{n}(1+\pi_{\tau_i}^i)^{N_t^i}$$

- The process ζ is non-negative if $\pi^i > -1$, for $i = 1, \ldots, n$
- The process ζ is a true martingale if $\mathbb{E}_{\mathbb{P}}[\zeta_t] = 1$ for any t or if π^i is bounded, for $i = 1, \ldots, n$

Hedging and martingale representation theorem

Theorem (Change of probability measure)

Define the probability measure $\mathbb{Q}$ *as*

$$d\mathbb{Q}|_{\mathcal{H}_t} = \zeta_t d\mathbb{P}|_{\mathcal{H}_t} .$$

where

$$\zeta_t = \exp\left(-\sum_{i=1}^{n}\int_0^t \pi_s^i \alpha_s^{i,\mathbb{P}} ds\right)\prod_{i=1}^{n}(1+\pi_{\tau_i}^i)^{N_t^i}$$

Then, for any $i = 1, \ldots, n$, *the process*

$$M_t^i := M_t^{i,\mathbb{P}} - \int_0^t \pi_s^i \alpha_s^{i,\mathbb{P}} ds = N_t^i - \int_0^t (1+\pi_s^i)\alpha_s^{i,\mathbb{P}} ds$$

is a $\mathbb{Q}$*-martingale. In particular, the* $(\mathbb{Q}, \mathbb{H})$*-intensity of* τ_i *is* $(1+\pi_t^i)\alpha_t^{i,\mathbb{P}}$.

Hedging and martingale representation theorem

- From the absence of arbitrage opportunity

$$\{\alpha_t^i > 0\} \overset{\mathbb{P}-a.s.}{=} \left\{\alpha_t^{i,\mathbb{P}} > 0\right\}$$

- For any $i = 1, \dots, n$, the process $\hat{\pi}^i$ defined by :

$$\hat{\pi}_t^i = \left(\frac{\alpha_t^i}{\alpha_t^{i,\mathbb{P}}} - 1\right)(1 - N_{t-}^i)$$

  is an $\mathbb{H}$-predictable process such that $\hat{\pi}^i > -1$

- The process ζ defined with $\pi^1 = \hat{\pi}^1, \dots, \pi^n = \hat{\pi}^n$ is an admissible Radon-Nikodym density
- Under $\mathbb{Q}$, credit spreads $\alpha^1, \dots, \alpha^n$ are exactly the intensities of default times

Hedging and martingale representation theorem

- The predictable representation theorem also holds under $\mathbb{Q}$
- In particular, if A is an $\mathcal{H}_T$ measurable payoff, then there exists $\mathbb{H}$-predictable processes $\hat{\theta}^i, i = 1, \dots, n$ such that

$$A = \mathbb{E}_{\mathbb{Q}}\left[A \mid \mathcal{H}_t\right] + \sum_{i=1}^{n} \int_t^T \hat{\theta}_s^i \underbrace{\left(dN_s^i - \alpha_s^i ds\right)}_{\text{CDS cash-flow}}.$$

- Starting from t the claim A can be replicated using the self-financed strategy with
  - the initial investment $V_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}A \mid \mathcal{H}_t\right]$ in the savings account
  - the holding of $\delta_s^i = \hat{\theta}_s^i e^{-r(T-s)}$ for $t \le s \le T$ and $i = 1, \dots, n$ in the instantaneous CDS
- As there is no charge to enter a CDS, the replication price of A at time t is $V_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}A \mid \mathcal{H}_t\right]$

Hedging and martingale representation theorem

- A depends on the default indicators of the names up to time T
  - includes the cash-flows of CDO tranches or basket credit default swaps, given deterministic recovery rates
- The latter theoretical framework can be extended to the case where actually traded CDS are considered as hedging instruments
  - See Cousin and Jeanblanc (2010) for an example with a portfolio composed of 2 names or in a general n-dimensional setting when default times are assumed to be ordered

Hedging and martingale representation theorem

- Risk-neutral measure can be explicitly constructed
  - We exhibit a continuous change of probability measure
- Completeness of the credit market stems from a martingale representation theorem
  - Perfect replication of claims which depend only upon the default history using CDS written on underlying names and default-free asset
  - Provide the replication price at time t
- But does not provide any operational way of constructing hedging strategies
- Markovian assumption is required to effectively compute hedging strategies

Markovian contagion model

- Pre-default intensities only depend on the **current status of defaults**

$$\alpha_t^i = \tilde{\alpha}^i \left(t, N_t^1, \ldots, N_t^n\right) 1_{t<\tau_i}, \; i = 1, \ldots, n$$

- Ex : Herbertsson - Rootzén (2006)

$$\tilde{\alpha}^i \left(t, N_t^1, \ldots, N_t^n\right) = a_i + \sum_{j \neq i} b_{i,j} N_t^j$$

- Ex : Lopatin (2008)

$$\tilde{\alpha}^i \left(t, N_t\right) = a_i(t) + b_i(t) f(t, N_t) \; \text{ with } N_t = \sum_{i=1}^{n} N_t^i$$

- Connection with continuous-time Markov chains
  - $\left(N_t^1, \ldots, N_t^n\right)$ Markov chain with possibly 2^n states
  - Default times follow a multivariate phase-type distribution

Homogeneous Markovian contagion model

- Pre-default intensities only depend on the current number of defaults
- All names have the same pre-default intensities $\tilde{\alpha}$

$$\alpha_t^i = \tilde{\alpha}\left(t, N_t\right) 1_{t<\tau_i}, \; i = 1, \ldots, n$$

where

$$N_t = \sum_{i=1}^{n} N_t^i$$

- This model is also referred to as the local intensity model

Homogeneous Markovian contagion model

- No simultaneous default, the intensity of N_t is equal to

$$\lambda(t, N_t) = (n - N_t)\tilde{\alpha}(t, N_t)$$

- N_t is a continuous-time Markov chain (pure birth process) with generator matrix :

$$\Lambda(t) = \begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & & 0 \\ & & \ddots & \ddots & \\ 0 & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Model involves as many parameters as the number of names

Homogeneous Markovian contagion model

Replication price of a European type payoff

$$V(t,k) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\Phi(N_T) \mid N_t = k\right]$$

- $V(t,k)$, $k = 0, \ldots, n-1$ solve the backward Kolmogorov differential equations :

$$\frac{\delta V(t,k)}{\delta t} = rV(t,k) - \lambda(t,k)\left(V(t,k+1) - V(t,k)\right)$$

- Approach also puts in practice by Schönbucher (2006), Herbersson (2007), Arnsdorf and Halperin (2007), Lopatin and Misirpashaev (2007), Cont and Minca (2008), Cont and Kan (2008), Cont, Deguest and Kan (2009)

Homogeneous Markovian contagion model

Computation of credit deltas

- $V(t, N_t)$, price of a CDO tranche (European type payoff)
- $V^I(t, N_t)$, price of the CDS index (European type payoff)

$$V(t, N_t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\Phi(N_T) \mid N_t\right]$$

$$V^I(t, N_t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\Phi^I(N_T) \mid N_t\right]$$

- Using standard Itô's calculus

$$dV(t, N_t) = \left(V(t, N_t) - \delta^I(t, N_t)V^I(t, N_t)\right) rdt + \delta^I(t, N_t)dV^I(t, N_t)$$

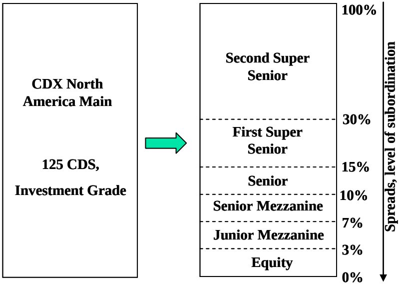
where

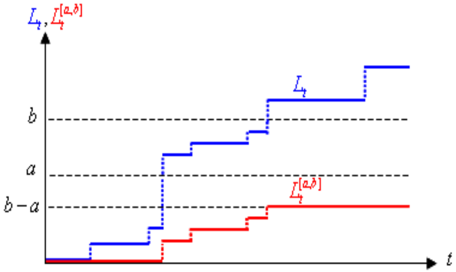
$$\delta^I(t, N_t) = \frac{V(t, N_t + 1) - V(t, N_t)}{V^I(t, N_t + 1) - V^I(t, N_t)}.$$

- Perfect replication with the index and the risk-free asset

CDO tranches on standard Index

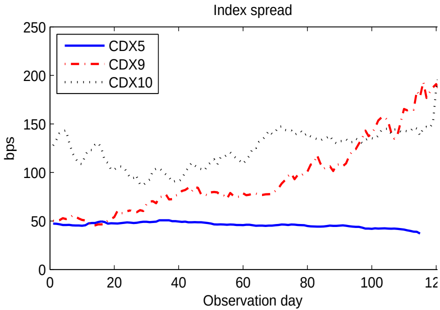
- Performance analysis of alternative hedging strategies developed for the correlation market
- CDO tranches on standard Index such as CDX North America Investment Grade index

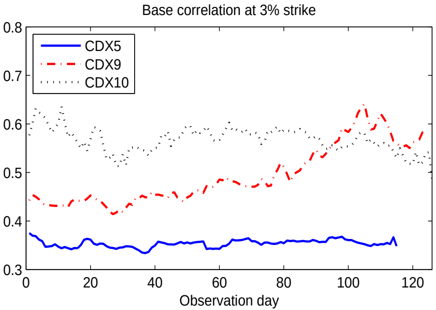




Data set

- Series 5 of the 5-year CDX NA IG from 20 September 2005 to 20 March 2006
- Series 9 of the 5-year CDX NA IG from 20 September 2007 to 20 March 2008
- Series 10 of the 5-year CDX NA IG from 21 March 2008 to 20 September 2008





Model Specifications

- **Gauss** : Base correlation approach based on the standard one-factor Gaussian copula pricing device
- **Para** : Local intensity model – parametric specification of local itensities

$$\lambda(t,k) = \lambda(k) = (n-k)\sum_{i=0}^{k} b_i$$

(Herbertsson (2008))

- **EM** : Local intensity model – local itensities $\lambda(t,k)$ obtained by minimizing a relative entropy distance with respect to a prior distribution

$$\inf_{\mathbb{Q}\in\Lambda} \mathbb{E}^{\mathbb{Q}_0}\left[\frac{d\mathbb{Q}}{d\mathbb{Q}_0}\ln\left(\frac{d\mathbb{Q}}{d\mathbb{Q}_0}\right)\right]$$

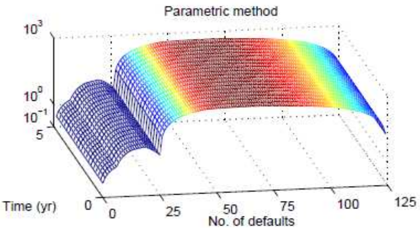
(Cont and Minca (2008))

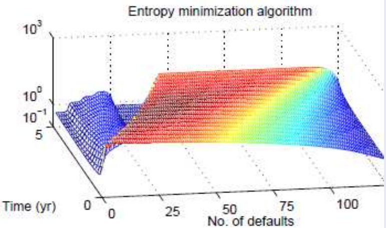
Calibration results

Root mean squared calibration errors (in percentage) :

	CDX5			CDX9			CDX10		
Tranche	Gauss	Para	EM	Gauss	Para	EM	Gauss	Para	EM
Index	0.04	5.15	5.14	0.03	4.40	4.81	0.02	6.73	6.77
0%-3%	0.01	2.35	2.36	0.00	1.31	1.32	0.01	1.69	1.68
3%-7%	0.00	0.51	0.69	0.00	0.61	0.86	0.00	1.04	1.03
7%-10%	0.00	0.08	1.32	0.00	0.24	0.91	0.00	0.43	0.39
10%-15%	0.00	0.06	1.77	0.00	0.24	1.15	0.00	0.40	0.36
15%-30%	0.00	0.29	1.97	0.01	1.19	1.74	0.01	1.80	1.68

Comparison of typical shapes of local intensities $\lambda(t,k)$, Para (left), EM (right)





Hedging ratios

Comparison of three alternative hedging methods

- Gauss delta : Index spread sensitivity computed in a one-factor Gaussian copula model calibrated at time t

$$\Delta_t^{\text{Gauss}} = \frac{\mathcal{V}(t, S_t + \varepsilon, \rho_t) - \mathcal{V}(t, S_t, \rho_t)}{\mathcal{V}^I(t, S_t + \varepsilon) - \mathcal{V}^I(t, S_t)}$$

  where $\mathcal{V}$ and $\mathcal{V}^I$ are the Gaussian copula pricing function associated with (resp.) the tranche and the CDS index. S_t is the Index spread at time t and ρ_t is the time-t base correlation.

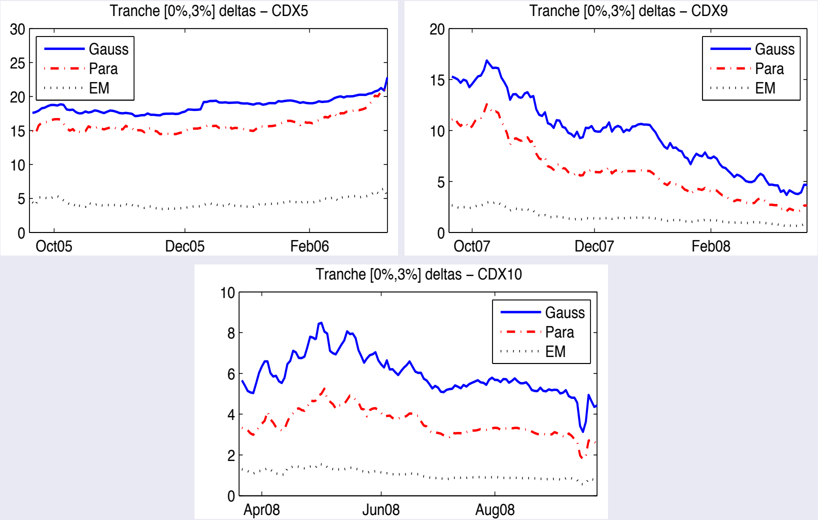
- Local intensity delta :

$$\delta^I(t, N_t) = \frac{V\,(t, N_t + 1) - V\,(t, N_t)}{V^I\,(t, N_t + 1) - V^I\,(t, N_t)}.$$

  with both Parametric (Param) and Entropy Minimisation (EM) calibration methods

Hedging ratios

Time series of equity tranche deltas, CDX.NA.IG series 5, 9 and 10



Hedging performance

Back-testing hedging experiments on series 5, 9 and 10 (1-day rebalancing)

$$\text{Relative hedging error} = \left| \frac{\text{Average P\&L increment of the hedged position}}{\text{Average P\&L increment of the unhedged position}} \right|,$$

$$\text{Residual volatility} = \frac{\text{P\&L increment volatility of the hedged position}}{\text{P\&L increment volatility of the unhedged position}}.$$

Relative hedging errors (in percentage)

	CDX5			CDX9			CDX10		
Tranche	Li	Para	EM	Li	Para	EM	Li	Para	EM
0%-3%	4	5	73	80	10	72	33	55	90
3%-7%	1	3	35	0.4	19	59	48	49	75
7%-10%	10	10	43	15	13	37	49	25	44
10%-15%	7	27	131	27	18	14	139	181	208
15%-30%	0.54	61	324	3	32	89	172	269	396

Hedging performance

Residual volatilities (in percentage)

	CDX5			CDX9			CDX10		
Tranche	Gauss	Para	EM	Gauss	Para	EM	Gauss	Para	EM
0%-3%	42	46	83	50	56	86	71	72	89
3%-7%	75	75	66	73	65	71	43	40	64
7%-10%	99	118	135	57	56	54	40	38	44
10%-15%	82	110	202	94	98	95	42	44	40
15%-30%	77	108	298	46	69	108	31	33	54

Conclusion :

- Hedging based on local intensity model with Entropy Minimisation calibration gives poor performance
- Before the crisis (CDX5), Gauss delta outperforms local intensity deltas
- During the crisis (CDX9 & CDX10), no clear evidence to discriminate between Gauss delta and Para local intensity delta

Conclusion

Thank you for your attention !

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