An extension of Davis and Lo’s contagion model

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Credit Risk, Systemic Risk, and Large Portfolios

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Empirical studies on contagion mechanisms

- Das and al. (2007) or Azizpour and Giesecke (2008): Conditional independence assumption with no contagion effect is rejected by historical default data. The conditional independence assumption is not enough to capture historical default dependency.

- Boissay (2006), Jorion and Zhang (2007, 2009) analyze the mechanism of default propagation and provide financial evidence of chain reactions or dominos effects.

Need for a dynamic model with defaults dependencies and contagion

- Eventual underlying macro-economic factors
- Contagion mechanisms
- Chain reactions and evolution over time
Some contagion models in the credit risk field

- Copula: Schönbucher and Schubert (2001)

In the spirit of Davis and Lo’s contagion model

- First models: Davis and Lo (2001)
- We propose a multiperiod extension of Davis and Lo’s contagion model.
Davis and Lo’s contagion model

Modeling of credit contagion for a pool of defaultable entities

- One-period model
- Credit references may default either directly or as a consequence of a contagion effect

Example: Portfolio with 5 credit references over one period

No direct default ($X_1=0$)

Direct default ($X_2=1$)

Contagion ($Y_{23}=1$)

No contagion ($Y_{24}=0$)
One-period contagion model

- \( n \) : number of credit references,
- \( X_i \) : direct default indicator of name \( i \) (i.e. \( X_i = 1 \) if \( i \) defaults directly, \( X_i = 0 \) otherwise),
- \( Y_{ji} = 1 \) if the contagion link is activated from name \( j \) to name \( i \), \( Y_{ji} = 0 \) otherwise.
- \( C_i \) : indirect default indicator of name \( i \),
- \( Z_i \) : global default indicator (direct or indirect) such that:

\[
Z_i = X_i + (1 - X_i)C_i
\]

where:

\[
C_i = \mathbf{1}_{\text{at least one } x_j Y_{ji} = 1, j=1,\ldots,n}
\]
Davis and Lo’s contagion model

\[ N = \sum_{i=1}^{n} Z_i : \text{total number of defaults} \]

**Distribution of total number of defaults (Davis and Lo)**

\[
P [N = k] = C_n^k \sum_{i=1}^{k} \binom{k}{i} p^i (1 - p)^{n-i} (1 - (1 - q)^i)^{k-i} (1 - q)^{i(n-k)}.
\]

**Under the assumptions:**

- Direct defaults \( X_i, i = 1, \ldots, n \) : iid Bernoulli with parameter \( p \)
- Contagion links \( Y_{ij}, i, j = 1, \ldots, n \) : iid Bernoulli with parameter \( q \)
- One contagion link alone may trigger an indirect default
- An infected entity cannot contaminate other entities (no chain-reaction effect)
Extension of Davis and Lo’s contagion model

**Domino Effect**

- The model becomes a multiperiod model
- One can choose the set of entities likely to contaminate others
- Some iid assumptions are released

\[ t=0 \quad t=1 \quad t=2 \]

...
Contagion incidence on indirect default

- One can change the number of contagions links required to cause a default (here two contaminations required)
Multi-period contagion model: \( t = 0, 1, 2, \ldots, T \), in period \([t, t + 1]\):

- \( n \): number of credit references,
- \( X_{it}^i \): direct default indicator of entity \( i \),
- \( Y_{ji}^{ii} \): contagion links are Bernoulli random variables such that \( Y_{ji}^{ii} = 1 \) if entity \( j \) may infect entity \( i \),
- \( Z_{it}^i \): default indicator (direct or indirect) such that:
  \[
  Z_{it}^i = Z_{it-1}^i + (1 - Z_{it-1}^i)[X_{it}^i + (1 - X_{it}^i)C_{it}^i]
  \]
  \( C_{it}^i = f\left(\sum_{j \in F_t} Y_{jt}^{ji}\right) \): indirect default indicator of name \( i \),
- \( F_t \) is the set of names that are likely to infect other names between \( t \) and \( t + 1 \)
- \( f \) is a contamination trigger function, for example \( f = \mathbb{1}_{x \geq 1} \) (Davis and Lo) or \( f = \mathbb{1}_{x \geq 2} \)
Extension of Davis and Lo’s contagion model

\[ N_t = \sum_{i=1}^{n} Z_t^i \]: total number of defaults at time \( t \)

**Main result**

\[
P[N_t = r] = \sum_{k=0}^{r} P[N_{t-1} = k] C_{n-k}^{r-k} \sum_{\gamma=0}^{r-k} C_{r-k}^\gamma \\
\cdot \sum_{\alpha=0}^{n-k-\gamma} C_{n-k-\gamma}^{\alpha} \mu_{\gamma+\alpha, t} \sum_{j=0}^{n-r} C_{n-r}^{j} (-1)^{j+\alpha} \xi_{j+r-k-\gamma, t(\gamma)}.
\]

**Under the assumptions:**

- \( X_t^i, i = 1, \ldots, n \) are conditionally independent Bernoulli r.v. with the same marginal distribution and \( X_t = (X_t^1, \ldots, X_t^n) \), \( t = 1, \ldots, T \) are independent vectors.
- \( Y_t^{ij}, i, j = 1, \ldots, n \) are conditionally independent Bernoulli r.v. with the same marginal distribution and \( Y_t = (Y_{t}^{ij})_{1 \leq i, j \leq n} \), \( t = 1, \ldots, T \) are independent vectors.
- \((X_t)_{t=1, \ldots, T}\) and \((Y_t)_{t=1, \ldots, T}\) are independent.
Similar kind of formulas hold when we have:

**finite-exchangeability**
- Direct defaults may be **finite-exchangeable** (does not imply conditional independence as infinite exchangeability, De Finetti’s Theorem does not apply here).

**evolution over time - non stationarity**
- Joint law for Direct defaults and for contagion links may change over time.

**heterogeneity (with higher complexity)**
- Direct defaults may be **dependent and heterogeneous**, in a monoperiodic framework.
- Direct defaults may be **dependent and heterogeneous**, in a multiperiodic framework, but with an exponential complexity (need to consider all possible sets of remaining entities at time $t$).
Waring’s Formula - special case of Schuette-Nesbitt Formula

If $B^1, \ldots, B^n$ are $n$ dependent Bernoulli r.v. and $\Gamma \subset \{1, \ldots, n\}$ with cardinal $m$, then

\[
P\left[ \sum_{i \in \Gamma} B^i = k \right] = \prod_{k \leq m} C^k_m \sum_{j=0}^{m-k} C^j_{m-k} (-1)^j \mu_{j+k}(\Gamma).
\]

with $\mu_k(\Gamma) = \frac{1}{C^k_m} \sum_{j_1 < j_2 < \ldots < j_k \in \Gamma} P\left[ B^{j_1} = 1 \cap \ldots \cap B^{j_k} = 1 \right]$, $k \geq 1$.

Coefficients $\mu_k$ may be simplified:

- if independence (without requiring iid): products
- if exchangeability: the sum vanishes

Here we are looking for:

- Directs defaults: $\sum_{j \in \Gamma} X^j_t$ as a function of some coefficients $\mu_{k,t}(\Gamma)$,
- Contagion links: $\sum_{j \in F_t} Y^{\sigma(j)}_t$ as a function of some coefficients $\lambda_{k,t}$,
- Indirects defaults: $\sum_{j=1}^{k} C^j_t$ as a function of some coefficients $\xi_{k,t}$,
Calibration on 5-years iTraxx tranche quotes

- Cash-flows of CDO tranches driven by the **aggregate loss process** (in %)

\[ L_t = \frac{1}{n} \sum_{i=1}^{n} (1 - R_i)Z_t^i \]

where \( R_i \) is the **recovery rate** associated with name \( i \).
Calibration on 5-years iTraxx tranche quotes

We restrict ourselves to the case where for all $t$:

- $X_t^i \sim \text{Bernoulli}(\Theta)$ where $\Theta \sim \text{Beta}$, $E[\Theta] = p$ and $\text{Var}(\Theta) = \sigma^2$, $i = 1, \ldots, n$
- $Y_t^{ij}$ are iid $Y_t^{ij} \sim \text{Bernoulli}(q)$, $i, j = 1, \ldots, n$
- Only one default is required to trigger a default by contagion

Moreover:

- $n = 125$, $r = 3\%$ (short-term interest rate)
- $R_i = R = 40\%$ for any $i = 1, \ldots, n$

$$L_t = \frac{1}{n}(1 - R) \cdot N_t$$

- Computation of CDO tranche price only requires marginal loss distributions at several time horizons
Least square calibration procedure: Find $\alpha^* = (p^*, \sigma^*, q^*)$ which minimizes:

$$RMSE(\alpha) = \sqrt{\frac{1}{6} \sum_{i=1}^{6} \left( \frac{\tilde{s}_i - s_i(\alpha)}{\tilde{s}_i} \right)^2}.$$ 

where

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<th>3%-6%</th>
<th>6%-9%</th>
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Calibration on 5-years iTraxx tranche quotes

Four calibration procedures:

- **Calibration 1**: All available market spreads are included in the fitting
- **Calibration 2**: The equity [0%-3%] tranche spread is excluded
- **Calibration 3**: Both equity [0%-3%] tranche and CDS index spreads are excluded
- **Calibration 4**: All tranche spreads are excluded except equity tranche and CDS index spreads.

Two calibration dates before and during the credit crisis:

- 31 August 2005
- 31 March 2008
## Calibration on 5-years iTraxx tranche quotes

### 31 August 2005

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### Annual scaled optimal parameters

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Calibration on 5-years iTraxx tranche quotes

31 March 2008

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Annual scaled optimal parameters

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Limits and perspectives

specificity of the model
- try to capture explicit microstructure of contagion
- contagion acts directly on random variables, not on probabilities
- one can say with certainty if default of entity $i$ is due to entity $j$

some limits of the model
- default rate depends on the number $n$ of entities
- contagions only within the considered portfolio
- numerical issues for large number $n$ of entities

some perspectives
- recursions to manage numerical issues
- contagions from outside the portfolio
- behavior when time tends to zero and $n$ becomes large
- asymptotic results - larger interconnected component
We propose a multi-period extension of Davis and Lo’s contagion model that accounts for

- possibly dominos or chain reaction effect
- flexible contagion mechanism (ex: more than one default required to trigger a contamination)
- explicitly model business interdependencies

We provide a recursive formula for the distribution of the number of defaults at different time horizons

- When direct defaults and contagion events are conditionally independent

The multi-period setting is required to price synthetic CDO tranches

- The contagion parameter has a significant impact on the model ability to fit CDO tranche quotes
I thank you for your attention.
Appendix I - probabilistic tools
### Infinite-exchangeability

A sequence of exchangeable r.v. if for all $n$ and for any permutation $\sigma$:

$$A_1, \ldots, A_n \overset{D}{=} A_{\sigma(1)}, \ldots, A_{\sigma(n)},$$

### De Finetti’s Theorem

A sequence of infinite-exchangeable Bernoulli r.v. is a sequence of infinite-exchangeable Bernoulli r.v. if and only if there exist a r.v. $\Theta \in [0,1]$ such that, conditionally to $\Theta$:

- $A_1, A_2, \ldots$ is an iid sequence of Bernoulli r.v. with parameter $\Theta$

- Here, calculations given $\Theta$ but difficulties to simplify

- De Finetti’s Theorem does not apply for finite-exchangeability

- Need for other tools
If $N$ is a number of fulfilled events $B_i$, $i \in \Omega$, 
A linear combination of $P[N = k]$ will be written:

### Schuette-Nesbitt formula

$$
\sum_{k \in \Omega} P[N = k] f(k) = \sum_{k \in \Omega} S_k \Delta^k f(0)
$$

avec $S_k = \sum_{j_1 < \ldots < j_k} P[B_{j_1} \cap \cdots \cap B_{j_k}]$

$$
\Delta f(k) = f(k + 1) - f(k), \text{ difference operator}
$$

- events of kind $[N = k]$ given coefficients $S_k$
- $S_k$ can be simplified with independence, without requiring i.i.d.
- $S_k$ can be simplified with exchangeability
- events of kind $[N = k]$ as simple as $[N = 0]$ or $[N \geq 1]$
Appendix I - Probabilistic tools

In the particular case where \( f(j) = \mathbb{1}_{j=k}, j \in \Omega \),

**Waring's formula**

If \( X^1_t, \ldots, X^n_t \) are \( n \) dependent Bernoulli r.v. and \( \Gamma \subset \Omega \) with cardinal \( m \),

\[
P \left[ \sum_{i \in \Gamma} X^i_t = k \right] = \mathbb{1}_{k \leq m} C^k_m \sum_{j=0}^{m-k} C^j_{m-k} (-1)^j \mu_{j+k, t}(\Gamma) .
\]

with

\[
\mu_{k, t}(\Gamma) = \frac{1}{C^k_{\text{card}(\Gamma)}} \sum_{j_1 < j_2 < \ldots < j_k} P \left[ X^{j_1}_t = 1 \cap \ldots \cap X^{j_k}_t = 1 \right], \quad k \geq 1,
\]

\[
\mu_{0, t}(\Gamma) = 1 \quad \text{even if } \Gamma = \emptyset .
\]
Interest in life-insurance framework:
- independence assumptions
- but different ages and non identically distributed lifetimes

Interest for Davis and Lo extension:
- one would like $P[N = k]$
- one can change more easily iid assumptions
- is simplified with exchangeability assumptions
Appendix I - Probabilistic tools

Idea from so-called Waring’s formula

for non iid Bernoulli r.v. $A_1, \ldots, A_n$, one can get the law of $\sum_j A_j$ as a function of coefficients of kind

$$P[A_1 = 1 \cap \cdots \cap A_i = 1].$$

- If independence: these coefficients become products
- If exchangeability: these coefficients does only depend on the number of considered r.v.

Here we are looking for:

- Directs defaults: $\sum_{j \in \Gamma} X^j_t$ as a function of coefficients $\mu_{k,t}(\Gamma)$,
- Contagion links: $\sum_{j \in F_t} Y^{(j)}_t$ as a function of coefficients $\lambda_{k,t}$,
- Indirects defaults: $\sum_{j=1}^{k} C^j_t$ as a function of coefficients $\xi_{k,t}$,
Appendix II - Basic numerical illustration
we consider here that for all $t$,

- $X_t^i$ are exchangeables, Bernoulli with hidden parameter $\Theta_X$, $E[\Theta_X] = p = 0.1$, $V[\Theta_X]$ is given
- $Y_{t}^{ij}$ are exchangeables, Bernoulli with hidden parameter $\Theta_Y$, $E[\Theta_Y] = q = 0.2$, $V[\Theta_Y]$ is given
- hidden parameters are Beta distributed

We consider

- 10 entities ($n = 10$),
- 10 temporal units ($T = 10$),
- average direct default probability $p = 0.1$,
- average contagion link probability $q = 0.2$. 
We define 4 models with common parameters:

1. **model 1**: $\sigma_X = 0$, $\sigma_Y = 0$, $f(x) = \mathbb{1}_{x \geq 1}$  
   (i.i.d. case, one contagion link required).

2. **model 2**: $\sigma_X = 0$, $\sigma_Y = 0$, $f(x) = \mathbb{1}_{x \geq 2}$  
   (i.i.d. case, two contagion links required).

3. **model 3**: $\sigma_X = 0.2$, $\sigma_Y = 0.2$, $f(x) = \mathbb{1}_{x \geq 1}$  
   (exchangeable case, one contagion link required).

4. **model 4**: $\sigma_X = 0.2$, $\sigma_Y = 0.2$, $f(x) = \mathbb{1}_{x \geq 2}$  
   (exchangeable case, two contagion link required).
Evolution of $E[N_t]$ as a function of $t$. i.i.d. case dotted.
Evolution of $V[N_t]$ as a function of $t$. i.i.d. case dotted.
Evolution of $P[N_t \geq 6]$ as a function of $t$. i.i.d. case dotted.
Evolution of $P[N_t \geq 10]$ as a function of $t$. i.i.d. case dotted.